1101 Analysis 1 Notes (Part 1 of 2)

Based on the 2011 autumn lectures by Dr C Wendl

MATH1101. Analysis 1

What is Analysis?

→ theory behind calculus (and beyond)

→ introduction to "serious" mathematics

focus on: AXIOMS, DEFINITIONS ⟶ PROPOSITIONS, LEMMAS, THEOREMS all of which require PROOF.

→ many things that appear straightforward at first ...... but are not!

$\boxed{\text{Ex}}$ 1. a sum of infinitely many numbers can (sometimes) be finite

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \sum_{i=1}^{\infty} 2^{-i} = 1$

but this requires a formal definition of the "=" sign.

however, it is also known that $\underbrace{(1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6})+(\frac{1}{7}-\frac{1}{8})+\cdots}_{\text{alternating harmonic series.}} = \ln 2$ (finite) — ①

multiplying by $\frac{1}{2}$ ...... $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}+\cdots = \frac{1}{2}\ln 2.$ — ②

spacing out ② ...... $\frac{1}{2} \quad - \frac{1}{4} \quad + \frac{1}{6} \quad - \frac{1}{8} \quad + \cdots = \frac{1}{2}\ln 2.$ — ③

from ①+③: ④ $\quad 1 + (\frac{1}{3}-\frac{1}{2}) + \frac{1}{5} + (\frac{1}{7}-\frac{1}{4}) + \frac{1}{9} + (\frac{1}{11}-\frac{1}{6}) = \frac{3}{2}\ln 2$?!

compare ① and ④ ... how is it that same sums (on the left) result in different values? problems with definitions

same sums in different orders ⟶ different ~~orders~~ results.

$\boxed{\text{Ex}}$ 2. Taylor series $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots\cdots$

consider $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ the function satisfies $f^{(n)}(0) = 0$ for all $n > 0$

⟶ $f(x) \neq$ its Taylor series $= 0 + 0x + 0x^2 + \cdots$ question: why not??

a problem of CONVERGENCE

$\boxed{\text{ADMIN!}}$ Resource page on Moodle: moodle.ucl.ac.uk Key for course: "analysis".

Textbook to procure: Binmore. *Mathematical Analysis* (2^{nd} ed.), cup. [approx £42]

* NUMBERS

$\mathbb{N}$	$\subset$	$\mathbb{Z}$	$\subset$	$\mathbb{Q}$	$\subset$	$\mathbb{R}$	$\subset$	$\mathbb{C}$
natural numbers		integers		rational numbers		real numbers		complex numbers (real + imaginary parts)

$\mathbb{N} = \{1, 2, 3, \ldots\}$: every natural number has a successor.

if $x \in \mathbb{N}$, then $x+1 \in \mathbb{N}$ also.

STRUCTURE OF $\mathbb{N}$.

① addition: if $x, y \in \mathbb{N}$, we can define $x+y \in \mathbb{N}$ ⟶ • operation is commutative $(x+y = y+x)$, associative $(x+y)+z = x+(y+z)$

② multiplication: if $x, y \in \mathbb{N}$, we can define $xy \in \mathbb{N}$ $(xy = yx)$ $(xy)z = x(yz)$.

③ order: for all [also written $\forall$] $x, y \in \mathbb{N}$, exactly one of the following is true:

(i) $x > y$ (ii) $x = y$ (iii) $x < y$. note: $x \leq y \overset{\text{if and only if}}{\Leftrightarrow} x < y$ or $x = y$.

$x \leq y$ is true $\Leftrightarrow$ $x > y$ is not true.

③-1. ordering is transitive: if $x, y, z \in \mathbb{N}$

$x < y$ and $y < z \Rightarrow x < z$ $\quad x \leq y$ and $y \leq z \Rightarrow x \leq z$

DRAWBACKS OF $\mathbb{N}$

the equation $x + n = m$, if $n \geq m$ has no solution $x \in \mathbb{N}$. ⟶ $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

PROPERTIES OF $\mathbb{Z}$ (in addition to traits shared by $\mathbb{N}$)

④ $\exists$ an <u>additive identity element</u> : $0 \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z}$, $0+x = x+0 = x$
$\exists$ an <u>additive inverse</u> : $\forall x \in \mathbb{Z}$, $\exists\, y \in \mathbb{Z}$ such that $x+y = y+x = 0$ (namely $y=-x$)
$\Big\} \Rightarrow (\mathbb{Z}, +)$ is a group.

...... we now can define <u>subtraction</u>

$$x, y \in \mathbb{Z},\quad x-y = x+(-y) \in \mathbb{Z}.$$

DRAWBACK OF $\mathbb{Z}$

the equation $mx=n$ has no solution $x \in \mathbb{Z}$ unless $m|n$. $\leadsto$ $\mathbb{Q} = \left\{\frac{p}{q} \,\middle|\, p,q \in \mathbb{Z},\ q \neq 0\right\}$
the set of <u>all</u> fractions $\frac{p}{q}$ <u>such that</u> $p,q \in \mathbb{Z}$ <u>and</u> $q \neq 0$.

note: if $\frac{p}{q} = \frac{m}{n} \not\Rightarrow p=m$ and $q=n$ e.g. $\frac{p}{q} = \frac{2p}{2q}$.

New PROPERTIES OF $\mathbb{Q}$

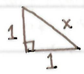
⑤ $\exists$ a <u>multiplicative identity element</u> : $1 \in \mathbb{Q}$ [also shared by $\mathbb{Z}$]
$\exists$ a <u>multiplicative inverse</u> : $\forall x = \frac{p}{q} \in \mathbb{Q}$ has, if $x \neq 0$ (i.e. $p \neq 0$), the inverse $x^{-1} = \frac{q}{p}$
$\Big\} \Rightarrow (\mathbb{Q}\setminus\{0\}, \cdot)$ is a group.

...... we now can define <u>division</u>

$$x = \frac{p}{q},\ y = \frac{m}{n} \in \mathbb{Q}\setminus\{0\},\quad \text{define } \frac{x}{y} = \frac{p}{q}\cdot\frac{n}{m} = \frac{pn}{qm} = xy^{-1}$$

? Question : Are the rational numbers "enough" to do serious mathematics?

- in some fields (e.g. number theory), perhaps ...
- but in some others (e.g. geometry), definitely not.



$1^2 + 1^2 = x^2 = 2$; $x \notin \mathbb{Q}$

$\boxed{\text{THEOREM}}$ 1. $\nexists$ any $x \in \mathbb{Q}$ such that $x^2 = 2$.

a <u>theorem</u> is an important statement that is <u>true</u> and it is <u>provable</u>

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a <u>lemma</u> is a less important theorem, helpful as a step in proving a more significant theorem.

LEMMA – if $n \in \mathbb{Z}$ is odd, then n^2 is odd i.e. if n^2 is even, then so is n. (contrapositive).

Proof: if n is odd, $n = 2k+1$ for some $k \in \mathbb{Z}$

$$\Rightarrow n^2 = (2k+1)^2 = 4k^2+4k+1 \equiv 1 \pmod 2.$$

thus n^2 is odd $\forall$ n is odd // q.e.d.

Proof of $\boxed{\text{THEOREM}}$ 1. By contradiction. $\longrightarrow$ assume claim is false; deduce something known to be false $\Rightarrow$ original claim must be true.

Assume claim is false i.e. $\exists$ some $x \in \mathbb{Q}$ s.t. $x^2 = 2$.

$\boxed{\text{CONCEPT}}$
WLOG: we can always arrange this assumption by making the right choices

then $x = \frac{p}{q}$ for some $p,q \in \mathbb{Z}$, $q \neq 0$.

assume, <u>WLOG</u>, that $\frac{p}{q}$ is simplified, so p and q are coprime; in particular then they are not both even.

$x^2 = 2 \Leftrightarrow \frac{p^2}{q^2} = 2 \Leftrightarrow p^2 = 2q^2 \Rightarrow p^2$ is even.

from the lemma, p^2 is even $\Rightarrow$ p is even $\therefore$ $\exists\, k \in \mathbb{Z}$ s.t. $p = 2k$

so $p^2 = 2q^2 \Leftrightarrow (2k)^2 = 2q^2 \Leftrightarrow 4k^2 = 2q^2 \Leftrightarrow 2k^2 = q^2 \Rightarrow q^2$ is even

from the lemma, q^2 is even $\Rightarrow$ q is even

thus p, q are both even, which contradicts the original assumption that p,q are not both even.

$\therefore$ original assumption is false, which implies $\nexists$ any $x \in \mathbb{Q}$ s.t. $x^2 = 2$ // q.e.d.

APPROXIMATION OF $\sqrt{2}$ WITH RATIONALS.

idea : Find $x^2 - 2y^2 = \pm 1$ where y is large, then $\frac{x^2}{y^2} - 2 = \pm\frac{1}{y^2} \Rightarrow \left(\frac{x}{y}\right)^2 = 2 \pm \frac{1}{y^2}$

we use a large y $\because$ $\lim_{y\to\infty} \frac{1}{y^2} = 0 \Rightarrow \lim_{y\to\infty}\left(\frac{x}{y}\right)^2 = \lim_{y\to\infty} 2 \pm \frac{1}{y^2} = 2$.

? How can we judge how close $\frac{x}{y}$ is to $\sqrt{2}$?

$\boxed{\text{Definition}}$ The absolute value (modulus) of x is $\to |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

In particular $|x| \geq 0$

and we interpret $|a-b|$ as the geometric distance between a and b.

Our problem reduces to : how can we estimate $\left|\frac{x}{y} - \sqrt{2}\right|$?

$$\left(\frac{x}{y}\right)^2 - 2 = \pm\frac{1}{y^2} \Rightarrow \left|\left(\frac{x}{y}\right)^2 - 2\right| = \frac{1}{y^2}$$

$$\left|\left(\frac{x}{y} - \sqrt{2}\right)\left(\frac{x}{y} + \sqrt{2}\right)\right| = \frac{1}{y^2}$$

$\boxed{\text{Lemma}}$ c.f. Binmore 1.16

$|a \cdot b| = |a||b|$

thus $\left|\frac{x}{y} - \sqrt{2}\right|\left|\frac{x}{y} + \sqrt{2}\right| = \frac{1}{y^2} \Rightarrow \left|\frac{x}{y} - \sqrt{2}\right|\left(\frac{x}{y} + \sqrt{2}\right) = \frac{1}{y^2}$ $\because \frac{x}{y} + \sqrt{2} > \sqrt{2} > 0$.

$\boxed{\text{Lemma}}$ if $a, b > 0$ and $a > b$, then $\frac{1}{a} < \frac{1}{b}$. [also $a \geq b$, then $\frac{1}{a} \leq \frac{1}{b}$].

Proof – Assume $a > b \Rightarrow \frac{1}{ab} \cdot a > \frac{1}{ab} \cdot b$ $(\because \frac{1}{ab} > 0)$

$\frac{1}{b} > \frac{1}{a}$

so $\left|\frac{x}{y} - \sqrt{2}\right| = \frac{1}{(\frac{x}{y} + \sqrt{2})} \cdot \frac{1}{y^2}$ and by the lemma, $\frac{1}{(\frac{x}{y} + \sqrt{2})} \cdot \frac{1}{y^2} < \frac{1}{\sqrt{2}} \cdot \frac{1}{y^2}$

and thus, $\left|\frac{x}{y} - \sqrt{2}\right| < \frac{1}{\sqrt{2}y^2}$ //

To approximate the value, we construct sequences of solutions x_n, y_n for $n = 1, 2, 3 \ldots$

s.t. (i) $x_1 = 1$, $y_1 = 1$

(ii) $x_{n+1} = x_n + 2y_n$; $y_{n+1} = x_n + y_n$

$\boxed{\text{Proposition}}$ (a minor theorem) $\forall n \in \mathbb{N}$, x_n, y_n defined in this way satisfy $x_n^2 - 2y_n^2 = \pm 1$

Proof – by induction

where $n = 1$, $x_n^2 - 2y_n^2 = x_1^2 - 2y_1^2 = 1 - 2(1) = 1 - 2 = -1$, the proposition is true.

assume the statement is true for some $k \in \mathbb{N}$ i.e. $x_k^2 - 2y_k^2 = \pm 1$ is true

then where $n = k+1$, $x_{k+1}^2 - 2y_{k+1}^2 = (x_k + 2y_k)^2 - 2(x_k + y_k)^2 = x_k^2 + 4x_ky_k + 4y_k^2 - 2x_k^2 - 4x_ky_k - 2y_k^2$

$= -x_k^2 + 2y_k^2 = -(x_k^2 - 2y_k^2) = \mp 1$ //

Since the statement is true for $n=1$ and $n=k$ is true $\Rightarrow$ $n=k+1$ is true ; the proposition is true $\forall n \in \mathbb{N}$ // q.e.d.

$\boxed{\text{Lemma}}$ $\forall n \in \mathbb{N}$, $x_n, y_n \in \mathbb{N}$ and $y_n \geq n$.

Proof – by induction

where $n = 1$, $x_1 = 1 \in \mathbb{N}$, $y_1 = 1 \in \mathbb{N}$, $y_1 \geq 1$, the proposition is true.

assume that for some k, $x_k, y_k \in \mathbb{N}$ and $y_k \geq k$.

$\rightarrow x_{k+1} = x_k + 2y_k \in \mathbb{N}$, $y_{k+1} = x_k + y_k \in \mathbb{N}$

also $y_{k+1} = x_k + y_k \geq 1 + k$ $(\because x_k \geq 1$ and $y_n \geq k)$

since the statement is true for $n = 1$ ..

"a theorem that follows from previous discussions"

$\boxed{\text{Corollary}}$ $\left|\frac{x_n}{y_n} - \sqrt{2}\right| < \frac{1}{\sqrt{2}n^2}$

Proof: $a \geq b \Rightarrow \begin{matrix} a^2 \geq ab \\ ab \geq b^2 \end{matrix} \Rightarrow a^2 \geq b^2$.

Proof – we showed that $\left|\frac{x}{y} - \sqrt{2}\right| < \frac{1}{\sqrt{2}} \cdot \frac{1}{y^2} \Rightarrow \left|\frac{x_n}{y_n} - \sqrt{2}\right| < \frac{1}{\sqrt{2}} \cdot \frac{1}{y_n^2}$

from our lemmata, $y_n \geq n \Rightarrow \frac{1}{y_n^2} \leq \frac{1}{n^2}$ and $\frac{1}{\sqrt{2}}\frac{1}{y_n^2} \leq \frac{1}{\sqrt{2}}\frac{1}{n^2}$ $(\because$ if $a, b > 0$ and $a \geq b$, then $a^2 \geq b^2)$

Calculating some values...

n	1	2	3	4	5	6	7	...
x_n	1	3	7	17	41	99	239	...
y_n	1	2	5	12	29	70	169	...
$\frac{x_n}{y_n}$	1	1.5	1.4	1.4167	1.4138	1.4129		$\cdots \longrightarrow \sqrt{2}$.

REVIEW OF THE NUMBERS...

$\mathbb{N} \underset{\text{subtraction}}{\subset} \mathbb{Z} \underset{\text{division}}{\subset} \mathbb{Q} \underset{\text{taking } \sqrt{2}}{\subset} \mathbb{R}$ real numbers!

a subset of $\mathbb{R}$ is $\mathbb{R}\backslash\mathbb{Q} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$, the irrational numbers

$\sqrt{2}, \sqrt{3}, \pi, e \in \mathbb{R}\backslash\mathbb{Q}$

of course, some irrational numbers are not as simple... unlike $\sqrt{2}$, which is simply the solution of $x^2-2=0$.

$\exists\ x \in \mathbb{R}$ that are not solutions to any polynomial equation with rational coefficients (transcendental numbers).

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REAL NUMBERS, $\mathbb{R}$.

$\boxed{\text{Definition}}$ A subset $S \subset \mathbb{R}$ is <u>bounded above</u> if $\exists\ H \subset \mathbb{R}$ (an upper bound from S) s.t. $\forall\ x \in S,\ x \leq H$.

A subset $S \subset \mathbb{R}$ is <u>bounded below</u> if $\exists\ h \subset \mathbb{R}$ (a lower bound from S) s.t. $\forall\ x \in S,\ x \geq h$.

$\boxed{\text{Ex.}}$ For some set $S = \{1,2,5\}$,

S is bounded above by $5, 6, 6.5, 30000 \ldots\ldots$ and bounded below by $1, 0, -2, -\pi, -1000000 \ldots\ldots$

<u>lub</u> $S = 5$ (least upper bound)   <u>glb</u> $S = 1$. (greatest lower bound)

$\boxed{\text{Ex.}}$ Take the set of $S = \{x \in \mathbb{R} \mid x > 0\}$

the set is unbounded above, bounded below by $0, -1, \cdots$ lub $S = 0$ (although it is not part of the set).

$\Longrightarrow$ S is an unbounded set.

$\boxed{\text{Proposition}}$ A subset $S \subset \mathbb{R}$ is bounded $\Longleftrightarrow$ $\exists\ H \geq 0$ s.t. $\forall x \in S,\ |x| \leq H$

note on proving IFF statements:
to prove $A \Leftrightarrow B$, then $\exists$ 2 things to prove
(i) $A \Rightarrow B$, and
(ii) $B \Rightarrow A$

Proof — we use the relation that $-|x| \leq x \leq |x|\ \forall\ x \in \mathbb{R}$

to prove: $\exists\ H \geq 0$ s.t. $\forall x \in S,\ |x| \leq H \Longrightarrow$ set is bounded

suppose it is true, then $-H \leq -|x| \leq x \leq |x| \leq H$

i.e. $-H$ is a lower bound, H is an upper bound $\longrightarrow$ S is bounded

to prove: subset $S \subset \mathbb{R}$ is bounded $\Longrightarrow \exists\ H \geq 0$ s.t. $\forall x \in S,\ |x| \leq H$

suppose S is bounded above and below, $\exists\ H \in \mathbb{R},\ h \in \mathbb{R}$ s.t. $\forall x \in S,\ h \leq x \leq H$.

we see that $-|H|-|h| \leq -|h| \leq h \leq x \leq H \leq |H| \leq |H|+|h|$

$\Longrightarrow -(|H|+|h|) \leq x \leq |H|+|h| \iff |x| \leq |H|+|h|$

since $|H|+|h| \geq 0$, the statement is proven, q.e.d.

$\boxed{\text{Ex.}}$ $S = \{x > 0 \mid x^2 < 2\}$ glb $S = 0$, lub $S = \sqrt{2}$

remark: if we only consider rational numbers, S would have no lub.

$\exists\ H \in \mathbb{Q}$ is an upper bound for S, then one can find a smaller $h \in \mathbb{Q},\ h < H$,

s.t. h is also an upper bound for S (HW2, problem 2(b))

CONTINUUM PROPERTY.

(i.e. why $\mathbb{R}$ is better than $\mathbb{Q}$).

For any non-empty subset $S \subset \mathbb{R}$,

(i) if S is bounded above, then it has a smallest upper bound : $\sup S \in \mathbb{R}$ ("supremum")

(ii) if S is bounded below, then it has a largest lower bound : $\inf S \in \mathbb{R}$ ("infimum").

Throughout this course, we will always assume that $\mathbb{R}$ has the following properties

(i) it contains $\mathbb{Q}$

(ii) it has the continuum property

(implication: $\mathbb{R}$ must contain $\sqrt{2}$, $\because \sqrt{2} = \sup \{x \in \mathbb{R} \mid x^2 < 2\}$)

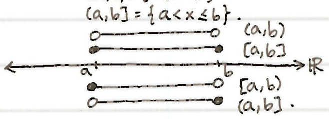
$\mathbb{Q}$ does not have the continuum property.

EXTREMA

$\boxed{\text{Definition}}$ $S \subset \mathbb{R}$ has maximum H (i.e. $\max S = H$) if $H \in S$ and H is an upper bound of S.

$S \subset \mathbb{R}$ has minimum h (i.e. $\min S = h$) if $h \in S$ and h is a lower bound of S.

note: intervals.
given $a<b$, we have
- open interval $(a,b) = \{a<x<b\}$
$= \{x \in \mathbb{R} \mid a<x<b\}$
- closed interval $[a,b] = \{a \le x \le b\}$
- half-closed / half open intervals
$[a,b) = \{a \le x < b\}$
$(a,b] = \{a < x \le b\}$.



$\boxed{\text{Ex.}}$ Given $[a,\infty) = \{x \ge a\}$, $(a,\infty) = \{x > a\}$

$(-\infty, b] = \{x \le b\}$, $(-\infty, b) = \{x < b\}$

$(-\infty, \infty) = \mathbb{R}$.

Find all $\inf S$, $\sup S$, $\max S$, $\min S$.

S	$\sup S$	$\inf S$	$\max S$	$\min S$
(a,b)	b	a	$\nexists$	$\nexists$
$[a,b]$	b	a	b	a
$[a,b)$	b	a	$\nexists$	a
$(a,b]$	b	a	b	$\nexists$
$[a,\infty)$	$\nexists$	a	$\nexists$	a
(a,∞)	$\nexists$	a	$\nexists$	$\nexists$
$(-\infty,b]$	b	$\nexists$	b	$\nexists$
$(-\infty,b)$	b	$\nexists$	$\nexists$	$\nexists$
$(-\infty,\infty)$	$\nexists$	$\nexists$	$\nexists$	$\nexists$

by examining $\sup S$ and $\inf S$, these intervals are unbounded.

note: $\nexists \sup S \Rightarrow \nexists \max S$, $\nexists \inf S \Rightarrow \nexists \min S$ but the reverse does not apply!

PROPERTIES OF BOUNDED SUBSETS

(1) Binmore 2.12. For any non-empty subset $S \subset \mathbb{R}$ bounded above, and any $c > 0$,

$$\sup_{x \in S} (cx) = c \sup_{x \in S} (x)$$

notation: for any function $F(x)$ defined for $x \in S$, we write $\sup_{x \in S} F(x) = \sup \{F(x) \mid x \in S\}$.
e.g. $\sup_{x \in S} x = \sup S$.

(2) *for homework, prove. For any non-empty subset $S \subset \mathbb{R}$ bounded above, and if $c > 0$,

$$\sup_{x \in S} (c+x) = c + \sup_{x \in S} (x)$$

let $T = \{c+x \mid x \in S\}$., since $x \le \sup_{x \in S} (x)$, $c > 0 \Rightarrow x + c \le \sup_{x \in S} (x) + c$; i.e. $\sup_{x \in S} (x) + c$ is an upper bound for $(c+x)$, and as $\sup_{x \in S} (c+x)$ is the lowest upper bound for $(c+x)$, $\sup_{x \in S} (x) + c \ge \sup_{x \in S} (c+x)$; similarly, $c + x \le \sup_{x \in S} (c+x) \Rightarrow x \le \underset{\text{(upper bound for } x)}{\sup_{x \in S} (c+x) - c} \Rightarrow \sup_{x \in S} (c+x) - c \ge \sup_{x \in S} (x)$

$\Rightarrow \sup_{x \in S} (c+x) \ge \sup_{x \in S} (x) + c$; thus $\sup_{x \in S} (x) + c = \sup_{x \in S} (c+x)$.

THE WELL-ORDERED PRINCIPLE

Every non-empty subset of $\mathbb{N}$ has a minimum. (we will consider this as an axiom of $\mathbb{N}$).

THE ARCHIMEDEAN PROPERTY.

$\boxed{\text{Theorem}}$ $\mathbb{N}$ is not bounded above. (or, $\nexists$ any $H \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $n \le H$).

Here we assume the following:

(i) every $n \in \mathbb{N}$ has a successor $n+1$

(ii) $\mathbb{R}$ contains $\mathbb{Q}$ and satisfies the continuum property (see pg 1101-005)

Proof of archimedian property — Proof by contradiction

Assume, conversely, that $\exists\, H \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $n \leq H$.

continuum property $\Rightarrow$ $\exists$ a least upper bound.

we assume, WLOG, H is the least upper bound, i.e. $H = \sup \mathbb{N}$ $\Rightarrow$

since $H-1 < H$, $H-1$ is not an upper bound for $\mathbb{N}$ $\Rightarrow$ $\exists\, n \in \mathbb{N}$ s.t. $n > H-1$. $\Rightarrow$

$(n+1) > H$, but $n+1 \in \mathbb{N}$ $\Rightarrow$ H is not an upper bound for $\mathbb{N}$.

This creates a contradiction as H was assumed to be an upper bound, $\therefore$ $\nexists$ an upper bound of $\mathbb{N}$.// q.e.d.

Corollary: $\nexists$ +ve real number that is smaller than every +ve rational number.

proof — In fact, for any given $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$; one can find a (large) natural number $n \in \mathbb{N}$ where $\frac{1}{n} \in \mathbb{Q}$, s.t. $\frac{1}{n} < \varepsilon$

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$\boxed{\text{Theorem}}$ The Principle of Induction.

Suppose $\forall n \in \mathbb{N}$, $P(n)$ denotes a statement (either true or false) involving the number n, and we know

(i) $P(1)$ is true, and

(ii) for all $n \in \mathbb{N}$, if $P(n)$ is true then so is $P(n+1)$;

Then $P(n)$ is true $\forall n \in \mathbb{N}$.

Proof — Define the subset $S = \{n \in \mathbb{N} \mid P(n) \text{ is not true}\}$

NTP: $S = \emptyset$

Proof by contradiction: Assume that $S \neq \emptyset$ $\Rightarrow$ by the well-ordering principle, S has a minimum, m such that $m \in S$.

since $m \in S$, and $P(m)$ is not true then $m \neq 1$ because $P(1)$ is true and $1 \notin S$.

$\Rightarrow m \geq 2$. $\Rightarrow$ $m-1 \in \mathbb{N}$.

since m is the minimum of S, $m-1 \notin S$. $\Rightarrow$ $P(m-1)$ is true.

but since $m-1 \in \mathbb{N}$, by hypothesis (ii), if $P(m-1)$ is true then so is $P(m-1+1) = P(m)$

and since $P(m)$ is true, $m \notin S$ which contradicts $m \in S$.

Thus the assumption $S \neq \emptyset$ does not hold, and $S = \emptyset$// q.e.d.

INDUCTION — more than just a theorem; is "too great an idea" to be packaged into a single theorem.

* $\boxed{\text{Ex}}$. Given $x_1, x_2, \ldots, x_n \in \mathbb{R}^+$.

we define the AM (arithmetic mean) A_n/ as $\frac{1}{n}\sum_{i=1}^{n} x_i$, and the GM (geometric mean) G_n/ as $\sqrt[n]{\prod_{i=1}^{n} x_i}$.

then prove the AM-GM inequality, where $G_n \leq A_n$

Proof — Let $P(n)$ denote the statement $G_n \leq A_n$ for some $n \in \mathbb{N}$.

We want to prove that this is true $\underline{\forall n} \in \mathbb{N}$, and we shall demonstrate that

(i) and (ii): induction by power of 2 $\longleftarrow$ (i) $P(2)$ is true, and (ii) $\forall n \in \mathbb{N}$, if $P(2^n)$ is true, then so is $P(2^{n+1})$, and

backward induction $\longleftarrow$ (iii) $\forall n \in \mathbb{N}$, $n \geq 2$, if $P(n)$ is true then so is $P(n-1)$.

Then since $\forall n \in \mathbb{N}$ $\exists$ a power of $2 > n$ $\Rightarrow$ $P(n)$ is true $\forall n \in \mathbb{N}$.

(i) $P(2)$ is true $\Leftrightarrow$ $\forall x_1, x_2 > 0$, $\sqrt{x_1 x_2} \leq \frac{x_1 + x_2}{2}$ because $0 \leq (\sqrt{x_1} - \sqrt{x_2})^2 \Rightarrow 2\sqrt{x_1 x_2} \leq x_1 + x_2$; hence hypothesis (i) holds. $A_2 \geq G_2$

(ii) let $n \in \mathbb{N}$, $m = 2^n$; then $2^{n+1} = 2^n \cdot 2 = 2m$. Then assuming $P(m)$ is true, we want to show that $P(2m)$ is true.

for any $x_1, x_2, \ldots, x_{2m}$; then NTP: $(x_1 x_2 \cdots x_{2m})^{\frac{1}{2m}} \leq \frac{x_1 + x_2 + \cdots + x_{2m}}{2m}$ if $(x_1 x_2 \cdots x_m)^{\frac{1}{m}} \leq \frac{x_1 + x_2 + \cdots + x_m}{m}$.

let $g_1 = (x_1 x_2 \cdots x_m)^{\frac{1}{m}}$ and $g_2 = (x_{m+1} x_{m+2} \cdots x_{2m})^{\frac{1}{m}}$; then $g_1 \leq \frac{x_1 + x_2 + \cdots + x_m}{m}$ and $g_2 \leq \frac{x_{m+1} + x_{m+2} + \cdots + x_{2m}}{m}$ (since $P(m)$ is true).

since $P(2)$ is true, $\sqrt[2]{g_1 g_2} \leq \frac{g_1 + g_2}{2}$ and $\sqrt[2m]{x_1 x_2 \cdots x_{2m}} \leq \frac{1}{2} \cdot \frac{x_1 + x_2 + \cdots + x_{2m}}{m}$ and $G_{2m} \leq A_{2m}$; hence hypothesis (ii) holds.

(iii) Assuming $P(n)$ is true, show $P(n-1)$; i.e. for any $x_1, x_2, \cdots, x_n > 0$ then

$$G_{n-1} = (x_1 x_2 \cdots x_{n-1})^{\frac{1}{n-1}} \leq \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} = A_{n-1} \quad \text{if} \quad G_n \leq A_n.$$

since $P(n)$ is true, if we let G_{n-1} be equal to term x_n, then

$$(x_1 x_2 \cdots x_{n-1} \cdot G_{n-1})^{\frac{1}{n}} \leq \frac{1}{n}(x_1 + x_2 + \cdots + x_{n-1} + G_{n-1})$$

$$\left[(x_1 x_2 \cdots x_{n-1})^{\frac{1}{n-1}}\right]^{\frac{n-1}{n}} \cdot G_{n-1}^{\frac{1}{n}} \leq \frac{1}{n}(x_1 + x_2 + \cdots + x_{n-1}) + \frac{1}{n} G_{n-1}$$

$$\left[(G_{n-1})^{n-1} \cdot G_{n-1}\right]^{\frac{1}{n}} \leq \left[\frac{1}{n-1}(x_1 + x_2 + \cdots + x_{n-1})\right]\left[\frac{n-1}{n}\right] + \frac{1}{n} G_{n-1}$$

$$\left[G_{n-1}^{\,n}\right]^{\frac{1}{n}} \leq \frac{(n-1)A_{n-1} + G_{n-1}}{n}$$

$$G_{n-1} \leq \frac{n-1}{n} A_{n-1} + \frac{1}{n} G_{n-1}$$

$$\frac{n-1}{n} G_{n-1} \leq \frac{n-1}{n} A_{n-1} \Rightarrow G_{n-1} \leq A_{n-1} \quad \because \frac{n-1}{n} > 0.$$

Hence, since (i) $P(2)$ is true, (ii) $P(2^n)$ is true $\Rightarrow P(2^{n+1})$ is true and (iii) $P(n)$ is true $\Rightarrow P(n-1)$ is true;

Then the statement applies for all $n \in \mathbb{N}$ // (since $n \geq 1$) q.e.d.

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Recall from Pg. 1: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = ?$

what does "=", the equality symbol, mean?

Let x_n = sum of first n terms $= \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n} \longrightarrow 1$ as $n \to \infty$.

$\boxed{\text{Ex}}$ How can be understand $0.333\cdots$?

consider successive approximations

$$0.3 = \frac{3}{10}, \quad 0.33 = \frac{33}{100}, \quad 0.333 = \frac{333}{1000} \longrightarrow 0.\dot{3} = \frac{1}{3} //$$

SEQUENCES.

$\boxed{\text{Definition}}$ A sequence of real numbers is an assignment to every $n \in \mathbb{N}$ of a real number $x_n \in \mathbb{R}$.

The range of a sequence in the set $\{x_n \mid n \in \mathbb{N}\}$.

The sequence is bounded above / bounded below / bounded $\Leftrightarrow$ its range is bounded.

$\boxed{\text{Ex}}$ • Examine the sequence $1, 2, 3, 4, \ldots$ i.e. $x_n = n$

It is bounded below, but not above.

• Or the sequence $999, 998, 997, \ldots$ i.e. $x_n = 1000 - n$.

It is bounded above, but not below

• Or the sequence $-1, 1, -1, 1, \cdots$ i.e. $x_n = (-1)^n$ is bounded.

Notation: The sequence as a whole can be expressed as

$\langle x_n \rangle$ or $\{x_n\}$ or (x_n) where $n \in \mathbb{N}$.

The individual n^{th} term is then expressed as x_n.

CONVERGENCE

$\boxed{\text{Definition}}$ A sequence $\langle x_n \rangle$ converges to a real number $l \in \mathbb{R}$ if the following holds:

for every $\varepsilon > 0$, $\exists N > 0$ s.t. $n > N \Rightarrow |x_n - l| < \varepsilon$

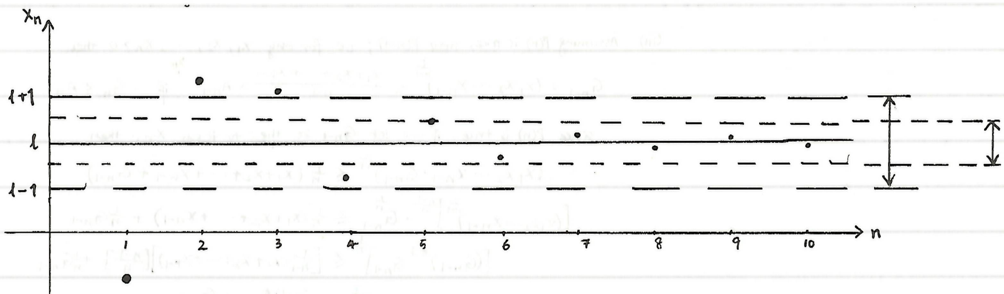
we say l is the limit of the sequence $\langle x_n \rangle$.

and we write $\lim_{n\to\infty} x_n = l$ or $\lim x_n = l$ or $x_n \to l$.

Remark – in particular, this definition must be true for arbitrarily small $\varepsilon > 0$,

then usually N must be very large

(depending on ε).



Given $\varepsilon=1$,

$n>3 \Rightarrow x_n \in (l-1, l+1) \iff |x_n - l|<1$

Given $\varepsilon=\frac{1}{2}$

$n>5 \Rightarrow x_n \in (l-\frac{1}{2}, l+\frac{1}{2}) \iff |x_n - l|<\frac{1}{2}$

* If $x_n \to l$, then we must be able to do this for arbitrarily small $\varepsilon>0$.

Proposition If $x_n \to l$, then $\langle x_n\rangle$ does not also converge to any other number $l' \neq l$

$\boxed{\text{Definition}}$ If $\langle x_n\rangle$ does not converge to any $l \in \mathbb{R}$, then we say $\langle x_n\rangle$ diverges.

$\boxed{\text{Ex}}$ Prove that $\frac{1}{n}$ converges to 0.

Let $x_n=\frac{1}{n}$, then given any $\varepsilon>0$,

we need to find $N>0$ s.t. $n>N \Rightarrow |x_n - 0| = \frac{1}{n}<\varepsilon$.

we have $\frac{1}{n}<\varepsilon \iff n>\frac{1}{\varepsilon}$ $\therefore$ it suffices to take $N=\frac{1}{\varepsilon}$, as then $n>N \iff n>\frac{1}{\varepsilon} \iff \frac{1}{n}<\varepsilon$, q.e.d.

Variation: Prove that for a constant $c \in \mathbb{R}$, $x_n = c+\frac{1}{n}$ converges to c.

Given any $\varepsilon>0$, we need $N>0$ s.t. $n>N \Rightarrow |x_n - c| = \frac{1}{n}<\varepsilon$

$\therefore$ again, $N=\frac{1}{\varepsilon}$ suffices, q.e.d.

$\boxed{\text{Ex}}$ If $x_n = c$, $c \in \mathbb{R}$, prove that $x_n \to c$

Given any $\varepsilon>0$, we need to find $N>0$ s.t. $n>N \Rightarrow |x_n - c|<\varepsilon$ whenever $n>N$.

But $|x_n - c| = 0 \ \forall n$, so this is true for any N, q.e.d.

$\boxed{\text{Ex}}$ For a sequence formed from the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$

let $\langle x_n\rangle$ be the sequence formed from the sum of the first n terms: i.e. $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$

Prove that $x_n \to 1$.

Note that $x_n = \frac{2^n-1}{2^n} = 1-\frac{1}{2^n}$

Given any $\varepsilon>0$, we need to find $N>0$ s.t. $n>N \Rightarrow |x_n - 1|<\varepsilon$,

$\divideontimes$ $|x_n-1| = \frac{1}{2^n} < \frac{1}{n}$ $\because$ $2^n>n \ \forall n$ (see Homework 2, Problem 1)

Now take $N=\frac{1}{\varepsilon}$, then $n>N \Rightarrow n>\frac{1}{\varepsilon} \Rightarrow \frac{1}{n}<\varepsilon \Rightarrow \frac{1}{2^n}<\frac{1}{n}<\varepsilon \Rightarrow |x_n - 1|<\varepsilon$, q.e.d.

Recall the sequences $\langle n\rangle$, $\langle 1000-n\rangle$, $\langle(-1)^n\rangle$; since they do not converge to some limit $l \in \mathbb{R}$,

these all diverge.   note that they do not get arbitrarily closer to any limit l.

$\boxed{\text{Ex}}$ It is claimed that if $x_n = \frac{1}{n^r}$ for any rational $r>0$, then $x_n \to 0$. Prove the claim.

NOTE: we restrict $r \in \mathbb{Q}$ because $r=\frac{p}{q}$ where $p,q \in \mathbb{N}$, then for $x>0$, $x^r = (x^p)^{1/q} = \sqrt[q]{x^p}$

Lemma: If $a,b>0$, then $a>b \iff a^n>b^n \ \forall n \in \mathbb{N}$, and $a \geq b \iff a^n \geq b^n$

Proof — we first prove $a>b \Rightarrow a^n>b^n$ and $a\geq b \Rightarrow a^n \geq b^n$ by induction

statement is true where $n=1$, assume truth for $n=k$ (i.e. $a>b \Rightarrow a^k>b^k$ and $a\geq b \Rightarrow a^{k+1} \geq b^{k+1}$)

if $a>b \Rightarrow a^k > b^k$, then $a^{k+1} = a\cdot a^k > a\cdot b^k$ and also $ab^k > bb^k = b^{k+1}$ $\because a>b$, thus a^k

(use same argument for $a \geq b \Rightarrow a^n \geq b^n$).

now, we need to prove $a^n > b^n \Rightarrow a>b$ and $a^n \geq b^n \Rightarrow a \geq b$

these statements imply the contrapositive "$\neg a>b \Rightarrow \neg a^n > b^n$" and "$\neg a \geq b \Rightarrow \neg a^n \geq b^n$"

$b \geq a \Rightarrow b^n \geq a^n$ and $b>a \Rightarrow b^n > a^n$ (which both have been proven earlier).

Given $\varepsilon > 0$, we need $N>0$ s.t. $n>N \Rightarrow |x_n - 0| = \frac{1}{n^r} < \varepsilon$

if $r = \frac{p}{q}$, $p,q \in \mathbb{N}$, then $\frac{1}{n^r} = \frac{1}{\sqrt[q]{n^p}} < \varepsilon \iff \sqrt[q]{n^p} > \frac{1}{\varepsilon} \underset{\text{(lemma)}}{\iff} n^p > \frac{1}{\varepsilon^q} \underset{\text{(lemma)}}{\iff} n > \frac{1}{\varepsilon^{p/q}}$

$\therefore$ it suffices to take $N = \varepsilon^{-\frac{q}{p}}$, q.e.d.

backwards, because $\frac{1}{p} \notin \mathbb{N}$ but $p \in \mathbb{N}$.

$\boxed{\text{Ex}}$ if $x \in \mathbb{R}$ st. $|x|<1$, prove that $x^n \to 0$.

Claim $\iff$ if $|x|>1$, then $\frac{1}{x^n} \to 0$.

Given $\varepsilon>0$, we need to find $N>0$ s.t. $n>N \Rightarrow \left|\frac{1}{x^n} - 0\right| = \frac{1}{|x|^n} < \varepsilon$, assuming $|x|>1$

write $h = |x|-1 > 0$, so $\frac{1}{|x|^n} = \frac{1}{(1+h)^n}$

by Bernoulli's inequality $[\, (1+h)^n \geq 1+nh \quad \forall n \in \mathbb{N},\ h \geq -1$: see Homework 2, Problem 1 $]$

$\therefore \frac{1}{|x|^n} = \frac{1}{(1+h)^n} \leq \frac{1}{1+nh}$ then $\frac{1}{1+nh} < \varepsilon \iff 1+nh > \frac{1}{\varepsilon} \iff nh = \frac{1}{\varepsilon} - 1 \iff n > \frac{1}{h}\left(\frac{1}{\varepsilon}-1\right)$

$\Rightarrow$ it suffices to take $N = \frac{1}{|x|-1}\left(\frac{1}{\varepsilon}-1\right)$, q.e.d.

$\boxed{\text{Ex}}$. $x_n = \frac{n^2-1}{n^2+1}$. Propose that $\langle x_n \rangle$ converges and prove it, finding its limit.

$$x_n = \frac{n^2-1}{n^2+1} = \frac{(n^{-2})(n^2-1)}{(n^{-2})(n^2+1)} = \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}} \to 1?$$

Given $\varepsilon > 0$, we need to find $N>0$ s.t. $n>N \Rightarrow |x_n - 1| = \left|\frac{n^2-1}{n^2+1} - \frac{n^2+1}{n^2+1}\right| = \left|-\frac{2}{n^2+1}\right| = \frac{2}{n^2+1} < \varepsilon \iff$

$\frac{n^2+1}{2} > \frac{1}{\varepsilon} \iff n^2+1 > \frac{2}{\varepsilon} \iff n^2 > \frac{2}{\varepsilon} - 1 \iff n > \sqrt{\frac{2}{\varepsilon}-1}$, $\therefore$ it suffices to take $N = \sqrt{\frac{2}{\varepsilon}-1}$, q.e.d.

"COMBINATION" THEOREM.

suppose $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences that converge, and $a, b \in \mathbb{R}$, then:

(i) $\lim (ax_n + by_n) = a \lim x_n + b \lim y_n$

(ii) $\lim (x_n y_n) = (\lim x_n)\cdot(\lim y_n)$

(iii) if $y_n \neq 0\ \forall n$ and $\lim y_n \neq 0$, then $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n}$

Using the theorem, the earlier example $x_n = \frac{n^2-1}{n^2+1}$ is reduced to

$$\lim\left(\frac{n^2-1}{n^2+1}\right) = \frac{\lim 1 - \frac{1}{n^2}}{\lim 1 + \frac{1}{n^2}} = \frac{\lim 1 - \lim \frac{1}{n^2}}{\lim 1 + \lim \frac{1}{n^2}} = \frac{1-0}{1+0} = 1$$

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An analyst's perspective — what is large? what is small?

"if ε is 'small enough', $1000\,\varepsilon$ is 'small enough'".

$\boxed{\text{Ex.}}$ $0.333\ldots$ is the limit of the sequence $\frac{3}{10}, \frac{33}{100}, \frac{333}{1000} \ldots$ Prove that it is $\frac{1}{3}$.

we define x_n as the difference of the term from $\frac{1}{3}$.

!? In general, $x_n = \frac{1}{3} - \frac{1}{3}\cdot\frac{1}{10^n} = \frac{1}{3}\left(1 - \frac{1}{10^n}\right) \longrightarrow \frac{1}{3}(1-0) = \frac{1}{3}$

OR. Given $\varepsilon>0$, we need to find $N>0$ s.t. $n>N \Rightarrow |x_n - 0| < \varepsilon \Rightarrow \left|\frac{1}{3}\left(1-\frac{1}{10^n}\right)\right| < \varepsilon \Rightarrow 1 - \frac{1}{10^n} < 3\varepsilon$

$\frac{1}{10^n} > 1 - 3\varepsilon \Rightarrow 10^n < \frac{1}{1-3\varepsilon} \Rightarrow n < \log_{10}\frac{1}{1-3\varepsilon}$

$\Rightarrow$ it suffices

$\boxed{\text{Ex.}}$ We know that $\frac{1}{n} \to 0$. Why does $\frac{1}{n}$ not converge to $\frac{1}{2}$?

Given any $\varepsilon>0$, can we find $N>0$ st. $\left|\frac{1}{n} - \frac{1}{2}\right| < \varepsilon$ $\underline{\forall}\, n > N$? NO, because beyond a certain $n = N_2 > N$, the condition fails for small ε.

$\boxed{\text{Ex.}}$ Both $x_n = n$ or $x_n = 1000 - n$ are sequences.

Both are unbounded.

$\boxed{\text{Theorem}}$ Any convergent sequence is ~~un~~bounded.

(and hence, all unbounded sequences diverge).

Proof – given any finite subset S of $\mathbb{R}$, $\max S$ and $\min S$ exists.

given also that $x_n \to \ell$, we know that for any $\varepsilon > 0$,

$\exists N > 0$ s.t. $|x_n - \ell| < \varepsilon \quad \forall n > N$.

WLOG, by increasing N if necessary, we may assume $N \in \mathbb{N}$.

$|x_n - \ell| < \varepsilon \Rightarrow -\varepsilon < x_n - \ell < \varepsilon \Rightarrow \ell - \varepsilon < x_n < \ell + \varepsilon$

since $\{x_1, x_2 \cdots, x_N\}$ is a finite set, it has a minimum and maximum.

$\therefore \min\{x_1, x_2, \dots, x_n, \ell - \varepsilon\} \leq x_n \leq \max\{x_1, x_2, \dots, x_n, \ell + \varepsilon\} \Rightarrow \langle x_n \rangle$ is bounded, q.e.d.

Proving the combination theorem.

Restatement of theorem. Assume $x_n \to x$ and $y_n \to y$, and $c \in \mathbb{R}$ is a constant. Then

(i-a) $cx_n \to cx$ and (i-b) $x_n + y_n \to x + y$

(ii) $x_n y_n \to xy$

(iii) if $x_n \neq 0$ and $x \neq 0$, $\frac{1}{x_n} \to \frac{1}{x}$

for if so, $\lim \frac{x_n}{y_n} \to \frac{x}{y}$

Proof – (reasoning) $x_n \to x \Rightarrow \forall \varepsilon > 0, \ \exists N > 0$ s.t. $n > N \Rightarrow |x_n - n| < \varepsilon$.

Then how far is cx_n from cx? $|cx_n - cx| = |c||x_n - x| < |c| \cdot \varepsilon = \varepsilon$

since ε is "small enough", so would $\frac{\varepsilon}{|c|}$ have been.

(official) (i-a)

if $c = 0$, then $cx_n = 0 \to 0 = cx$.

Now assume $c \neq 0$, since $x_n \to x$, $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $n > N \Rightarrow |x_n - x| < \frac{\varepsilon}{|c|}$

Then $n > N \Rightarrow |cx_n - cx| = |c||x_n - x| < |c|\frac{\varepsilon}{|c|} = \varepsilon$.

Hence $cx_n \to cx$, q.e.d.

(i-b)

$x_n \to x \Rightarrow \forall \varepsilon > 0, \ \exists N_1 > 0$ st. $n > N_1 \Rightarrow |x_n - n| < \varepsilon/2$

$y_n \to y \Rightarrow \forall \varepsilon > 0, \ \exists N_2 > 0$ s.t. $n > N_2 \Rightarrow |y_n - y| < \varepsilon/2$

Hence $n > \max\{N_1, N_2\}$, $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

triangle inequality

Thus, $x_n + y_n \to x + y$, q.e.d.

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(ii)

since x_n converges, it is bounded i.e. $\exists C > 0$ s.t. $|x_n| \leq C \ \forall n$.

Given $\varepsilon > 0$, since $x_n \to x$ and $y_n \to y$, $\exists N_1, N_2 > 0$ st. $n > N_1 \Rightarrow |x_n - x| < k_\varepsilon = \frac{1}{C + |y|}\varepsilon$, and

$n > N_2 \Rightarrow |y_n - y| < k_\varepsilon = \frac{1}{C + |y|}\varepsilon$.

Let $N = \max\{N_1, N_2\}$, so if $n > N$, then $|x_n - x|, |y_n - y| < \frac{1}{C + |y|}\varepsilon$.

Thus, $|x_n y_n - xy| = |x_n(y_n - y) + y(x_n - x)| \leq |x_n(y_n - y)| + |y(x_n - x)| = |x_n||y_n - y| + |y||x_n - x|$

$< C\frac{\varepsilon}{C + |y|} + |y|\frac{\varepsilon}{C + |y|} = \varepsilon$,

hence, $x_n y_n \to xy$, q.e.d.

(iii)

since x_n converges, it is bounded i.e. $\exists C > 0$ s.t. $|x_n| \leq C \ \forall n$

sequence is bounded away from 0.

sequence is non-zero

caution $|x_n| \geq C > 0 \not\Leftrightarrow |x_n| > 0$ for all n.

Lemma – if $\langle x_n \rangle$ is a sequence with $x_n \neq 0 \ \forall n$ and $x_n \to x \neq 0$, then $\exists C > 0$ s.t. $|x_n| \geq C \ \forall n$.

Proof – for sufficiently large $N > 0$, $n > N \Rightarrow |x_n - x| < \frac{|x|}{2} \Rightarrow |x_n| = |x + (x_n - x)| \geq |x| - |x - x_n| = \frac{|x|}{2}$.

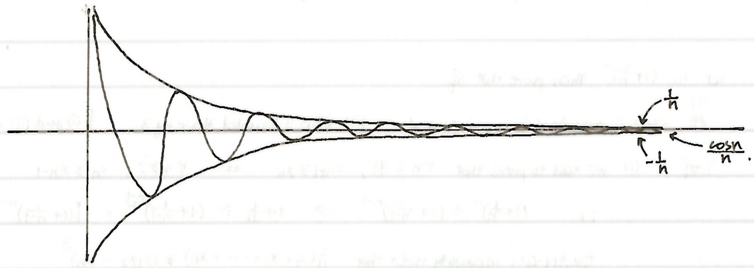
and $\forall n$, $|x_n| \geq \min\{x_1, x_2, \dots, x_n, \frac{|x|}{2}\}$.

recall Δ inequality:
$|a + b| \ngeq |a| + |b|$
$\Leftrightarrow$
$|a + b| + |b| \ngeq |a|$
$\Leftrightarrow$
$|(a + b) - b| \leq |a + b| + |-b|$

Given $\varepsilon > 0$, since $x_n \to x$, $\exists N > 0$ s.t. $n > N \Rightarrow |x_n - x| <^{|x|}$

Ex Prove $\frac{\cos n}{n} \to 0$.

since $|\cos n| \le 1 \Rightarrow -1 \le \cos n \le 1 \Rightarrow -\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$ as $n \to \infty$.



SANDWICH / SQUEEZE THEOREM.

Suppose $\langle x_n \rangle, \langle y_n \rangle, \langle z_n \rangle$ are sequences s.t. $x_n \le y_n \le z_n \ \forall n$, and

$\exists l \in \mathbb{R}$ s.t. $x_n \to l$ as $z_n \to l$, then $y_n \to l$ as well.

Proof – since $x_n \to l$ and $z_n \to l$, for any given $\varepsilon > 0$ $\exists$ numbers $N_1, N_2 > 0$ s.t.

$n > N_1 \Rightarrow |x_n - l| < \varepsilon$ and $n > N_2 \Rightarrow |z_n - l| < \varepsilon$

let $N = \max\{N_1, N_2\}$ and assume $n > N$; then $-\varepsilon < x_n - l < \varepsilon \Rightarrow l - \varepsilon < x_n < l + \varepsilon$

$-\varepsilon < z_n - l < \varepsilon \Rightarrow l - \varepsilon < z_n < l + \varepsilon$.

$\therefore$ $l - \varepsilon < x_n \le y_n \le z_n < l + \varepsilon \Rightarrow l - \varepsilon < y_n < l + \varepsilon \Rightarrow |y_n - l| < \varepsilon$, which proves that $y_n \to l$, q.e.d.

Ex For $x > 0$, consider $\langle x^{\frac{1}{n}} \rangle$ for $n \in \mathbb{N}$, i.e. $\langle x^{\frac{1}{n}} \rangle = \{x, \sqrt[2]{x}, \sqrt[3]{x}, \ldots\}$. show that $x^{\frac{1}{n}} \to 1$.

Use the AM-GM inequality for the n positive numbers $y_1 = y_2 = \cdots = y_{n-1} = 1$, $y_n = x$.

then $(y_1 y_2 \ldots y_n)^{\frac{1}{n}} \le \frac{y_1 + y_2 + \ldots + y_n}{n} \Rightarrow x^{\frac{1}{n}} \le \frac{n-1+x}{n} = 1 + \frac{x-1}{n}$

thus. $x^{\frac{1}{n}} - 1 \le \frac{x-1}{n}$

suppose $x \ge 1$, then $x^{\frac{1}{n}} \ge 1 \ \forall n$, so $0 \le x^{\frac{1}{n}} - 1 \le \frac{x-1}{n}$, and as $n \to \infty$,

$0 \le \lim_{n \to \infty} x^{\frac{1}{n}} - 1 \le 0$

by the sandwich theorem, $x^{\frac{1}{n}} - 1 \to 0$ and $x^{\frac{1}{n}} \to 1$

suppose $0 < x < 1$, then $\frac{1}{x} > 1$, so by the argument above, $\frac{1}{x^{1/n}} = (\frac{1}{x})^{\frac{1}{n}} \to 1$.

and by the combination theorem, $x^{\frac{1}{n}} = (\frac{1}{x^{1/n}})^{-1} = [(\frac{1}{x})^{\frac{1}{n}}]^{-1} \to 1^{-1} = 1$

thus, $x^{\frac{1}{n}} \to 1$ for all $x > 0$, q.e.d.

MONOTONE SERIES.

Definition $\langle x_n \rangle$ is increasing if $\forall n$, $x_{n+1} \ge x_n$ and strictly increasing if $\forall n$, $x_{n+1} > x_n$

$\langle x_n \rangle$ is decreasing if $\forall n$, $x_{n+1} \le x_n$ and strictly decreasing if $\forall n$, $x_{n+1} < x_n$.

for all these cases, $\langle x_n \rangle$ is monotone.

Ex Both $\langle \frac{1}{n} \rangle$ and $\langle -\frac{1}{n} \rangle$ are both monotone (increasing / decreasing).

$\langle \frac{(-1)^n}{n} \rangle$ is not monotone, neither is $\langle \frac{\cos n}{n} \rangle$, nor $\langle (-1)^n \rangle$, which does not even converge, although it is bounded.

Theorem If $\langle x_n \rangle$ is increasing and bounded above, then it converges to its smallest upper bound $\sup \langle x_n \rangle$.

If $\langle x_n \rangle$ is decreasing and bounded below, then it converges to its greatest lower bound $\inf \langle x_n \rangle$.

Proof – let $l = \sup\{x_n \mid n \in \mathbb{N}\}$. Then for any $\varepsilon > 0$, l is an upper bound for $\{x_n\}$, but $l - \varepsilon$ is not $\because \exists N \in \mathbb{N}$ st. $x_N > l - \varepsilon$

Then for $n>N$, sequence $\langle x_n \rangle$ is increasing $\Rightarrow$ $x_n \geq x_N > l-\varepsilon$, but since l is an upper bound, $x_n \leq l$.

$\therefore$ $l-\varepsilon < x_N \leq x_n \leq l < l+\varepsilon$ $\Rightarrow$ $|x_n - l| < \varepsilon$, and the proof is complete // q.e.d. proof for $\inf \langle x_n \rangle$ is analogous.

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sometimes, it is sufficient to prove existence of something rather than try to calculate it:

$\boxed{\text{Ex}}$ let $x_n = (1+\frac{1}{n})^n$. Then prove that

(i) $\langle x_n \rangle$ is monotone increasing, and (ii) $\langle x_n \rangle$ is bounded above by 3. [(i) and (ii) imply that $\langle x_n \rangle$ converges to a number ≤ 3 (actually e)]

Proof — (i) we need to prove that $\forall n \in \mathbb{N}$, $x_{n+1} \geq x_n$ $\iff$ $\forall n \geq 2$, $x_n \geq x_{n-1}$

i.e. $(1+\frac{1}{n})^n \geq (1+\frac{1}{n-1})^{n-1}$ $\Rightarrow$ $1+\frac{1}{n} \geq (1+\frac{1}{n-1})^{\frac{n-1}{n}} = [(1+\frac{1}{n-1})^{n-1}]^{\frac{1}{n}}$

the AM-GM inequality states that $\frac{1}{n}(x_1+x_2+\cdots+x_n) \geq (x_1x_2\cdots x_n)^{\frac{1}{n}}$

let $x_1 = x_2 = \cdots = x_{n-1} = 1+\frac{1}{n-1}$ and $x_n = 1$, then $\frac{1}{n}[(n-1)(1+\frac{1}{n-1})+1] \geq [(1+\frac{1}{n-1})^{n-1}(1)]^{\frac{1}{n}}$

$1+\frac{1}{n} \geq (1+\frac{1}{n-1})^{\frac{n-1}{n}}$ // q.e.d.

(ii) Using the binomial theorem, $(1+\frac{1}{n})^n = 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2!}(\frac{1}{n})^2 + \cdots + (\frac{1}{n})^n = 1+1+(1-\frac{1}{n})\frac{1}{2!} + (1-\frac{1}{n})(1-\frac{2}{n})\frac{1}{3!} + \cdots + (1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-$

$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} = 1+\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 1+2[1-(\frac{1}{2})^n] < 3$ //

RECURSIVE SEQUENCES.

$\boxed{\text{Ex}}$ Assume $x_1 = \frac{1}{2}$, and define the subsequent terms recursively by

$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}$ (by induction, we can show that $x_n = \frac{2^n-1}{2^n}$).

This sequence converges to 1.

$\boxed{\text{Ex}}$ choose any $a>0$ constant, and any $x_1>0$, then $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$.

if $a=2$, $x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2}$, then it can be proven that $\lim x_n = \sqrt{2}$. (specific case).

$\boxed{\text{Definition}}$ If $\langle x_n \rangle$ is any sequence and $\langle j_n \rangle$ is a strictly increasing sequence of natural numbers,

then the sequence $\langle y_n \rangle$ defined such that:

$y_n = x_{j_n}$ is a <u>subsequence</u> of $\langle x_n \rangle$.

for instance, subsequences of $\langle x_n \rangle$ include $\langle x_{n+1} \rangle$, $\langle x_{2n} \rangle$, $\langle x_{2^n-1} \rangle$, ...

$\boxed{\text{Theorem}}$ If $x_n \to l$, then every subsequence of $\langle x_n \rangle$ also converges to l.

Proof — depends on fact that for any strictly increasing $\langle j_n \rangle$, $n \in \mathbb{N}$, we have $j_n \geq n$ $\forall n \in \mathbb{N}$.

then we use $x_n \to l$ as basis for proof of convergence.

$\boxed{\text{Ex}}$ $\langle (-1)^n \rangle$ is bounded, but still does not converge.

Working — If we define $x_n = (-1)^n$, we can identify two subsequences:

$\langle x_{2n} \rangle = \langle 1 \rangle$ and $\langle x_{2n-1} \rangle = \langle -1 \rangle$. both subsequences converge, but to different limits $\Rightarrow$ sequence does not converge.

$\boxed{\text{Ex}}$ (cont'd) Consider where, in general, $x_1 > 0$, $x_{n+1} = \frac{x_n + \frac{a}{x_n}}{2}$, $a>0$.

assume for the moment that $\langle x_n \rangle$ converges and its limit l exists, i.e. $x_n \to l \in \mathbb{R}$.

then $\langle x_{n+1} \rangle$ must also converge to the same limit.

this gives us $\lim x_{n+1} = \lim \frac{x_n + \frac{a}{x_n}}{2} = \lim x_n$

i.e. $l = \frac{l+\frac{a}{l}}{2}$, and if $l \neq 0$ $\Rightarrow$ $2l = l + \frac{a}{l}$ $\Rightarrow$ $l^2 = a$.

since $l^2 = a$, $l = \sqrt{a}$ (proving by induction that since $x_1 > 0$, $l = \lim_{n\to\infty} x_n > 0$, hence we reject $-\sqrt{a}$).

this tells us that if the limit exists and $l \neq 0$, then $l = \sqrt{a}$

but how do we know that the limit actually exists?

claim 1: $\forall n \geq 2$, $x_n \geq \sqrt{a}$ and claim 2: the sequence $\langle x_{n+1} \rangle$ is decreasing.

proof of 1 – $x_n \geq \sqrt{a} \Rightarrow x_{n+1} \geq \sqrt{a}\ \forall n \geq 1$

recall that $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$ and by the AM-GM inequality,

$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}) \geq \sqrt{x_n \cdot \frac{a}{x_n}} = \sqrt{a} \Rightarrow$ claim 1 is true.

proof of 2 – need to show $x_{n+1} \leq x_n$ i.e. $\frac{1}{2}(x_n + \frac{a}{x_n}) \leq x_n \Rightarrow x_n + \frac{a}{x_n} \leq 2x_n \Rightarrow$

$\frac{a}{x_n} \leq x_n \Rightarrow a \leq x_n^2 \Rightarrow x_n \geq \sqrt{a}$, which reduces to claim 1.

since claim 1 is true, claim 2 is true.

so, since claims 1 and 2 are true, the sequence is monotone decreasing to a greatest lower bound, and the limit exists.

Theorem: If $x_n \geq a$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = l \Rightarrow l \geq a$

If $x_n \leq b$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = l \Rightarrow l \leq b$.

Proof – see details in 1101P-003.

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What is Analysis?

Analysis is the study of convergence.

Q: what conditions on a sequence guarantee that it converges? example, a monotone bounded sequence.

but being bounded alone is not enough, such as the oscillating series $\langle (-1)^n \rangle$, which has 2 subsequences which converge to different limits.

BOLZANO – WEIERSTRASS THEOREM.

"Every bounded sequence has a convergent subsequence."

Proof – we have established that if a sequence is bounded and monotone, it must converge, which follows from

Theorem: Every sequence has a monotone subsequence.

Proof – we call $n \in \mathbb{N}$ a peak point if $\forall m > n$, $x_n \geq x_m$.

we distinguish 2 cases:

(1) if there are infinitely many peak points $k_1, k_2, \ldots \in \mathbb{N}$.

then the subsequence $\langle x_{k_n} \rangle$ is monotone decreasing.

(2) if there are finitely many peak points $k_1, k_2, \ldots, k_N \in \mathbb{N}$.

then for any $n > k_N$, $\exists\ n' > n$ s.t. $x_{n'} > x_n$

$\therefore$ we can always construct an increasing subsequence $\langle x_{j_n} \rangle$ with $j_n > k_n\ \forall n$, q.e.d.

thus, if this subsequence is monotone, and since the sequence is bounded $\Rightarrow$ the subsequence is bounded, then the subsequence must converge, q.e.d.

Ex: $\langle (-1)^n \rangle$ is a bounded sequence, and thus it contains convergent sequences that tend to 1 and -1.

Ex: Take all fractions from 0 to 1: i.e. $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \ldots$

calling this sequence $\langle x_n \rangle$, its range contains every $r \in \mathbb{Q}$ with $0 < r < 1$.

i.e. $\langle x_n \rangle$ is bounded by 0 and 1, but does not converge.

$$\left.\begin{array}{l} \text{subsequence 1: } \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \longrightarrow 0 \\ \text{subsequence 2: } \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \longrightarrow 1 \end{array}\right\} \text{the sequence must diverge; but not just that...}$$

Claim: For any $l = [0,1]$, $\langle x_n \rangle$ has a subsequence convergent to l.

Proposition ("$\mathbb{Q}$ is DENSE in $\mathbb{R}$"): Given any real numbers $a < b$, $\exists\ r \in \mathbb{Q}$ s.t. $a < r < b$.

Proof of proposition — First assume $a, b > 0$; then $\exists\ q \in \mathbb{N}$ s.t. $q > \frac{1}{b-a}$ (by the Archimedean property).

$\Leftrightarrow \frac{1}{q} < b-a$. Let $p = \min\{n \in \mathbb{N} \mid \frac{n}{q} > a\}$. (again by the Archimedean property, we choose n).

the set is non-empty, $\therefore$ by the well-ordered principle, min {set} exists and p exists.

thus $\frac{p}{q} > a$, and since $\frac{p}{q}$ is the smallest number greater than a, $\frac{p-1}{q} \leq a \Rightarrow$

$$\frac{p}{q} = \frac{p-1}{q} + \frac{1}{q} < a + (b-a) = b.$$

and letting $r = \frac{p}{q}$, $a < r < b$, q.e.d.

Proof of claim — we have already found subsequences $\to 0$ and 1, so we assume $l \in (0,1)$.

also, every rational number in $(0,1)$ occurs in $\langle x_n \rangle$ infinitely many times (non-simplified versions).

for instance, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$ and $\frac{p}{q} = \frac{pk}{qk}$ $\forall\ k \in \mathbb{N}$.

$\therefore$ we can find a subsequence $\langle x_{j_n} \rangle$ s.t. $x_{j_n} \in (l - \frac{1}{n}, l + \frac{1}{n})$.

from the proposition, $\exists\ r \in \mathbb{Q}$ s.t. $l - \frac{1}{n} < r < l + \frac{1}{n}$, and r occurs infinitely many times in the sequence.

Also, since it is sometimes difficult to compute the limit of a sequence,

we seek a notion of convergence without mentioning the limit.

convergence $\xrightarrow{\text{idea}}$ successive terms in the sequence get closer together.

$\boxed{\text{Ex}}$ Take $x_n = \frac{1}{n}$.

for $N > 0$, if $n > N$ and $m > N$, then $|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \overset{\text{triangle inequality}}{\leq} \frac{1}{n} + \frac{1}{m} < \frac{2}{N} \Rightarrow$ can be made arbitrarily small as N increases.

On the other hand, take $x_n = (-1)^n$.

then $|x_{2n} - x_{2n-1}| = |1 - (-1)| = 2$, which does not get small for large n?

Question: Is it enough to recognise that $|x_{n+1} - x_n| \to 0$ as $n \to \infty$? NO.

$\boxed{\text{Ex}}$ Take $x_n = \sqrt{n}$. Then $\langle x_n \rangle$ diverges to $+\infty$, but

$$|x_{n+1} - x_n| = |\sqrt{n+1} - \sqrt{n}| = \left|(\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right| = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0 \text{ as } n \to \infty.$$

Hence, it is not true that $|x_n - x_m|$ is always small when both m and n are large; only small when successive.

CAUCHY SEQUENCES.

$\boxed{\text{Definition}}$ A sequence $\langle x_n \rangle$ is called a Cauchy sequence if for every $\varepsilon > 0$,

$\exists\ N > 0$ s.t. $m, n > N \Rightarrow |x_n - x_m| < \varepsilon$.

$\boxed{\text{Ex}}$ $\langle \frac{1}{n} \rangle$ is Cauchy.

$\boxed{\text{Proposition}}$ $\langle x_n \rangle$ is convergent $\Rightarrow$ $\langle x_n \rangle$ is Cauchy.

Proof — given $\varepsilon > 0$, $\exists\ N > 0$ s.t. $n > N \Rightarrow |x_n - \lim x_n| < \frac{\varepsilon}{2}$.

then if $m, n > N$, $|x_n - x_m| = |x_n - \lim x_n + \lim x_n - x_m| \overset{(\Delta)}{\leq} |x_n - \lim x_n| + |x_m - \lim x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, q.e.d.

COMPLETENESS OF $\mathbb{R}$ (also the "general principle of convergence")

$\boxed{\text{Theorem}}$ A sequence converges $\Leftrightarrow$ it is Cauchy.

Proof –
we have already proven the forward statement, we hence only need to show that Cauchy $\Rightarrow$ convergence.

lemma: a Cauchy sequence is bounded.

Proof – choose $\varepsilon = 1$, then if $\langle x_n \rangle$ is Cauchy, $\exists\ N > 0$ s.t. $m, n > N \Rightarrow |x_n - x_m| < 1$.

then in particular, $|x_n - x_{N+1}| < 1\ \forall\ n > N \Rightarrow |x_n - x_{N+1} + x_{N+1}| = |x_n| \leq |x_n - x_{N+1}| + |x_{N+1}|$.

hence, $|x_n| < 1 + |x_{N+1}|$ $\therefore \forall n$, $|x_n| \leq \max\{|x_1|, |x_2|, \ldots, |x_N|, 1 + |x_{N+1}|\}$. $\Rightarrow$ bounded, q.e.d.

Assume $\langle x_n \rangle$ is Cauchy, then by the lemma states that $\langle x_n \rangle$ is bounded, and by the Bolzano-Weierstrass theorem, it has a convergent subsequence.

subsequence: $\langle x_{n_j} \rangle$, $x_{n_j} \to l \in \mathbb{R}$ $\therefore$ given $\varepsilon > 0$, $\exists\ R > 0$ s.t. $j > R \Rightarrow |x_{n_j} - l| < \frac{\varepsilon}{2}$, and since sequence is Cauchy, $\exists N > 0$, $m, n > N \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}$.

Take $n>N$ and $k>R$ large enough s.t. $n_K > N$ also, then

$$|x_n - \ell| = |x_n - x_{n_K} + x_{n_K} - \ell| \le |x_n - x_{n_K}| + |x_{n_K} - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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Adding real numbers......

What are the real numbers?

ARCHITECTURE OF $\mathbb{R}$.

$\mathbb{R}$ is...... (1) unlimited/unending decimal expansions — but then what is $x+y$?

(2) "points on a line" $\longleftarrow \overset{0}{|} \overset{x}{\times} \longrightarrow \mathbb{R}$. — but how thick is a "point"?

(3) thus far in the course, we have assumed that $\mathbb{R}$ is a set containing $\mathbb{Q}$, and has 2 algebraic operations "+" (addition) and "·" (multiplication), which satisfy axiomatic properties (A1) to (A9). [of a field]. However, (A1)–(A9) also can define $\mathbb{Q}$...

(A1) $(x+y)+z = x+(y+z)$ $\forall x,y,z \in \mathbb{R}$.

(A2) $\exists\, 0 \in \mathbb{R}$ s.t. $0+x = x$ $\forall x \in \mathbb{R}$

(A3) $\forall x \in S$ $\exists -x \in S$ s.t. $x+(-x)=0$ → allows us to define subtraction.

(A4) $x+y = y+x$ $\forall x,y \in \mathbb{R}$.

(A5) $x(yz) = (xy)z$ $\forall x,y,z \in \mathbb{R}$.

(A6) $\exists\, 1 \in \mathbb{R}$ s.t. $1 \cdot x = x$ $\forall x \in \mathbb{R}$.

(A7) $\forall x \in \mathbb{R}$ with $x \ne 0$, $\exists\, x^{-1} \in \mathbb{R}$ s.t. $x(x^{-1}) = 1$ → allows us to define division.

(A8) $xy = yx$ $\forall x,y \in \mathbb{R}$.

(A9) $x(y+z) = xy + yz$ $\forall x,y,z \in \mathbb{R}$

so we include two more axioms...

(O1) there exists an ordering relation ">" s.t. $\forall x \in \mathbb{R}$, either $x>0$, $x=0$ or $x<0$.

(O2) $\forall x,y \in \mathbb{R}$, if $x,y>0$ then $x+y>0$ and $xy>0$.

from this, we have $\boxed{\text{definition}}$ $x>y \iff x-y>0$.

claim — transitivity: $x>y$ and $y>z \Rightarrow x>z$.

Proof - we assume $x-y>0$ and $y-z>0$, then $x-y+y-z>0 \Rightarrow x>z$, q.e.d.

but (O1)–(O2) are also shared by $\mathbb{Q}$; so we add the continuum property. (C)

(C). THE CONTINUUM PROPERTY:

Every non-empty subset of $\mathbb{R}$ bounded from above/below has a supremum/infimum respectively.

⇒ bounded monotone sequences converge ⇒ bounded sequences have convergent subsequences (Bolzano-Weierstrass Thm)

⇒ Cauchy sequences converge.

Question: Does any set having all these properties actually exist?

Recall: We have showed that if $\mathbb{R}$ exists, then $\mathbb{Q}$ are "dense" in $\mathbb{R}$ (i.e. every $x \in \mathbb{R}$ can be approximated arbitrarily well by natural numbers, i.e. $\forall x \in \mathbb{R}$, $\exists$ a sequence $\langle x_n \rangle$ s.t. $x_n \in \mathbb{Q}$ $\forall n$ and $x_n \to x$.

⟶ idea: we define $\mathbb{R}$ to be convergent sequences of $\mathbb{Q}$.

but to do this without assuming existence of $\mathbb{R}$, we cannot state a limit $\ell \in \mathbb{R}$, so we use rational Cauchy sequences:

$\langle x_n \rangle$, $x_n \in \mathbb{Q}$ s.t. $\forall \varepsilon>0$ with $\varepsilon \in \mathbb{Q}$, $\exists N>0, N \in \mathbb{Q}$ s.t. $n,m>N \Rightarrow |x_n - x_m| < \varepsilon$.

$\boxed{\text{Definition}}$ $\mathbb{R} = \{\text{rational Cauchy sequences}\}$

For $\langle x_n \rangle, \langle y_n \rangle \in \mathbb{R}$, we define $\langle x_n \rangle + \langle y_n \rangle = \langle x_n + y_n \rangle$ and $\langle x_n \rangle \cdot \langle y_n \rangle = \langle x_n y_n \rangle$.

we identify each $r \in \mathbb{Q}$ with the constant sequence $x_n = r$.

Define $\langle x_n \rangle > 0$ iff $\exists N>0$ and $\delta>0$, $\delta \in \mathbb{Q}$ s.t. $x_n \ge \delta$ $\forall n>N$.

Question: Does our definition of $\langle x_n \rangle + \langle y_n \rangle$ and $\langle x_n \rangle \cdot \langle y_n \rangle$ make sense?

Lemma - if $\langle x_n \rangle, \langle y_n \rangle$ are Cauchy, then so are $\langle x_n + y_n \rangle$ and $\langle x_n y_n \rangle$.

Proof for sum — we know $\forall \varepsilon>0$, $\exists N>0$ s.t. $|x_n - x_m| < \varepsilon$, $n,m>N$ and also $\exists N'>0$ s.t. $|y_n - y_m| < \varepsilon$, $n,m>N'$

now if $m, n > \max\{N, N'\}$, $|(x_n+y_n)-(x_m+y_m)| = |(x_n-x_m)+(y_n-y_m)| \leq |x_n-x_m| + |y_n - y_m|$

which is arbitrarily small by choosing ε small enough $\Rightarrow$ $\langle x_n+y_n \rangle$ is Cauchy.

we "identify" 2 sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ if $x_n - y_n \to 0$ $\rightsquigarrow \mathbb{R}$ (equivalence classes!).

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INFINITE SERIES. (i.e. sums).

series: $\sum_n a_n$ or $\sum_{n=1}^{\infty} a_n$.

$\boxed{\text{Ex}}$ Zeno's paradox. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$?

$\boxed{\text{Ex}}$ $0 = 0+0+0+0+\ldots = (1-1)+(1-1)+(1-1)+\cdots = 1-1+1-1+1-1+\cdots = 1+(-1+1)+(-1+1)+\cdots = 1 \ldots$ $0=1$??!! no.

$\boxed{\text{Definition}}$ The partial sums of a series $\sum_n a_n$ are the numbers $S_N = \sum_{n=1}^{N} a_n$.

These form a sequence $S_1, S_2, S_3, \ldots$

We write $\sum_n a_n = l$ and say that the series converges to $l \iff S_N \to l$ as $l \to \infty$.

$\boxed{\text{Proposition}}$ if $\sum_n a_n$ converges, then

(i) $a_n \to 0$ (as a sequence)

(ii) $\sum_{n=N}^{\infty} a_n \to 0$ as $N \to \infty$.

(iii) For any $c \in \mathbb{R}$, $\sum_n c a_n = c \sum_n a_n$.

(iv) if $\sum_n b_n$ also converges, then $\sum (a_n + b_n) = \sum_n a_n + \sum_n b_n$.

Proof of proposition (i) — $S_N = \sum_{n=1}^{\infty} a_n \Rightarrow S_N - S_{N-1} = a_n$; then $l - l = 0$, q.e.d.

(iii) — if $\sum_{n=1}^{\infty} a_n = l$, then $\sum_n c a_n = \lim_{N\to\infty} \sum_{n=1}^{N} c a_n$ (finite sum) $= \lim_{n\to\infty} c \sum_{n=1}^{N} a_n$ (distributive law for finite series).

$= c \lim_{N\to\infty} \sum_{n=1}^{\infty} a_n = c \lim_{N\to\infty} \sum_n^{N} a_n = c \sum_n a_n$ // q.e.d.

$\boxed{\text{Ex}}$ (geometric series) Given $x \in \mathbb{R}$, what is $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$?

$x^n \to 0 \iff |x| < 1 \Rightarrow \sum_n x^n$ can converge $\iff |x| < 1$ (otherwise it diverges).

assume $|x| < 1$, then $S_N = 1 + x + x^2 + \cdots + x^N$

$S_{N+1} = 1 + x + x^2 + \cdots + x^N + x^{N+1} = S_N + x^{N+1} = 1 + x S_N$.

$\Rightarrow 1 - x^{N+1} = S_N (1-x) \Rightarrow S_N = \frac{1-x^{N+1}}{1-x} \to \frac{1}{1-x}$ as $N \to \infty$ //.

$\boxed{\text{Ex}}$ (telescoping series). $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$

trick: use partial fractions — $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

then $S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{N-1}-\frac{1}{N}\right) + \left(\frac{1}{N}-\frac{1}{N+1}\right) = 1 - \frac{1}{N+1}$

The Taylor series of an infinitely differentiable function $f(x)$ at $x=0$ is

$f(x) \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ e.g. for $f(x) = \frac{1}{1-x} \Rightarrow f^n(0) = n! \Rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

this is true, but only when $|x| < 1$; otherwise it always diverges.

Question: Given a series $\sum_n a_n$, does it converge or diverge? (often cannot be computed).

Proposition — if $a_n \geq 0 \; \forall n$, then $\sum_n a_n$ converges $\iff$ partial sums are bounded above (since $\langle S_N \rangle$ is monotone increasing).

i.e. $\exists H$ s.t. $\forall N \in \mathbb{N}$, $\sum_{n=1}^{N} a_n \leq H$.

Proof — $\langle S_n \rangle$ is monotone increasing, $\therefore$ converges $\iff$ bounded above, q.e.d.

Impt! Sometimes we cannot compute limit of series — e.g. Riemann-Zeta function.

$\boxed{\text{Ex}}$ (i) $\sum_n \frac{1}{n^p}$ for $p > 1$, $p \in \mathbb{Q}$. $\zeta(p)$ $\longrightarrow$ converges.

(ii) $\sum \frac{1}{n}$ $\zeta(1)$. $\longrightarrow$ diverges! (note: just because $a_n \to 0$, it does not guarantee $\sum_n a_n$ converges!)
only converse is true.

relatedly, $\int_1^{\infty} \frac{1}{x^p} dx$ is finite when $p > 1$, but infinite when $p = 1$.

Ex. if $0 < p < 1$, then $\sum_n \frac{1}{n^p}$ diverges.

Proof — $n^p \leq n \Rightarrow \frac{1}{n^p} \geq \frac{1}{n} \Rightarrow \sum_{n=1}^{N} \frac{1}{n^p} \geq \sum_{n=1}^{N} \frac{1}{n}$

by contradiction: if $\sum \frac{1}{n^p}$ converged, its partial sums would be bounded above $\Rightarrow$ partial sums of $\sum_n \frac{1}{n}$ also bounded above $\Rightarrow$ $\sum \frac{1}{n}$ converges (contradiction!) thus, $\sum \frac{1}{n^p}$ diverges.

Theorem COMPARISON TEST.

if $0 \leq a_n \leq b_n \ \forall n$ and $\sum_n b_n$ converges; then $\sum_n a_n$ also converges.

Proof — same as in proof that $\sum_n \frac{1}{n^p}$ diverges.

Ex. Does $\sum_n \frac{1}{n^3-2}$ converge? (we expect so, since $\sum_n \frac{1}{n^3}$ converges).

(Remark) we cannot directly apply the comparison test as stated above, because.

(i) $n=1$ term is negative

(ii) $\frac{1}{n^3-2} \leq \frac{1}{n^3}$ is not true; but! at least $\frac{1}{n^3-2} \leq \frac{2}{n^3}$ where $n \geq 2$ ($\because \frac{2}{2n^3-4} \leq \frac{2}{n^3} \Rightarrow n^3-4 \geq 0$).

lemma! — $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=N}^{\infty} a_n$ also converges.

i.e. removing finitely many terms does not affect convergence, but the sum may be different.

Proof of lemma —

so, $\forall n \geq 2$, $0 \leq \frac{1}{n^3-2} \leq \frac{2}{n^3}$; and $\sum_n \frac{2}{n^3}$ converges $\Rightarrow$ by comparison test, $\sum_{n=2}^{\infty} \frac{1}{n^3-2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^3-2}$ converges, q.e.d.

What if there are also negative terms?

Ex. $\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+\dots$ has partial sums $1, 0, 1, 0, 1, 0, \dots$

are bounded but not monotone $\Rightarrow$ not convergent.

Definition A series $\sum_n a_n$ converges absolutely $\Leftrightarrow$ $\underbrace{\sum_n |a_n|}_{\text{all terms +ve.}}$ converges; $\Leftrightarrow$ partial sums $\sum_{n=1}^{N} |a_n|$ are bounded above.

Ex. $\sum_n \frac{(-1)^n}{n^2}$ converges absolutely since $\sum_n \left|\frac{(-1)^n}{n^2}\right| = \sum_n \frac{1}{n^2}$ converges.

Theorem $\sum_n a_n$ converges absolutely $\Rightarrow \sum_n a_n$ converges.

Proof — write $S_N = \sum_{n=1}^{N} a_n$. say $N > M$, both are large, then

$|S_N - S_M| = \left|\sum_{n=M+1}^{N} a_n\right| \leq \sum_{n=M+1}^{N} |a_n|$ (triangle inequality).

if M, N are large; since $\sum_n a_n$ converges absolutely, $\sum_{n=M+1}^{N} |a_n|$ converges. i.e. $|S_N - S_M| < \varepsilon$.

$\Rightarrow \langle S_n \rangle$ is a Cauchy sequence, $\therefore$ it converges, q.e.d.

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recall: $\sum_n a_n$ has partial sums $S_N = \sum_{n=1}^{N} a_n$; and $\sum_{n=1}^{\infty} a_n = l \Leftrightarrow S_N \to l$ as $N \to \infty$. (convergent series).

special case: $a_n \geq 0 \ \forall n \Rightarrow \langle S_n \rangle$ is monotone increasing:

$\therefore$ converges iff bdd above: either $S_N \to l \geq 0$ for some l; or

diverges to $+\infty$ if $S_N \to +\infty$; we write $\sum_n a_n = +\infty$.

Ex. Prove that $\sum_n (-1)_n$ diverges i.e. $\sum_n \frac{1}{n} = \infty$.

(we need to show partial sums are unbounded).

Proof — for any $N \in \mathbb{N}$, consider the 2^{N}-th partial sum, $S_{2^N} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^N} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \dots + (\frac{1}{2^{N-1}+1} + \dots + \frac{1}{2^N})$

hence, $S_{2^N} > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + \dots + (\frac{1}{2^N} + \dots + \frac{1}{2^N}) = 1 + \frac{1}{2} + 2(\frac{1}{4}) + 4(\frac{1}{8}) + \dots + N(\frac{1}{2^N}) = 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}}_{N \text{ terms}}$

$= 1 + N(\frac{1}{2}) = 1 + \frac{N}{2}$.

as $N \to \infty$, $\frac{N}{2} \to \infty$ and $S_{2^N} \to \infty \Rightarrow \langle S_{2^N} \rangle$ is not bounded $\Rightarrow \langle S_n \rangle$ is not bounded.

$\sum_n \frac{1}{n}$ diverges, and since $\frac{1}{n} > 0 \ \forall n \in \mathbb{N}$; $\sum_n \frac{1}{n} = +\infty$.

claim: if $p>1$, $\sum_n \frac{1}{n^p} < \infty$

Proof – we need to show partial sums are bounded. since $\langle S_N \rangle$ is increasing, if we find a subsequence that is bounded above, it suffices $\Rightarrow$ whole sequence also is bounded above.

consider $S_{2^N-1} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2^N-1)^p} = 1 + \left(\frac{1}{2^p}+\frac{1}{3^p}\right) + \left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right) + \cdots + \left(\frac{1}{(2^{N-1})^p} + \cdots + \frac{1}{(2^N-1)^p}\right)$

if $p>1$, $m>n \Rightarrow m^p > n^p \Rightarrow \frac{1}{m^p} < \frac{1}{n^p}$

thus, $S_{2^N-1} < 1 + 2\left(\frac{1}{2^p}\right) + 4\left(\frac{1}{4^p}\right) + 8\left(\frac{1}{8^p}\right) + \cdots + 2^{N-1}\frac{1}{(2^{N-1})^p} = 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \cdots + \frac{1}{(2^{N-1})^{p-1}}$

$= \sum_{n=0}^{N-1} \frac{1}{(2^{p-1})^n} = \frac{1-\left(\frac{1}{2^{p-1}}\right)^N}{1-\frac{1}{2^{p-1}}} \longrightarrow \frac{1}{1-\frac{1}{2^{p-1}}}$ as $N \to \infty$.

$\therefore$ bounded $\Rightarrow$ S_{2^N-1} is also bounded above, and $\langle S_N \rangle$ is bounded above $\Rightarrow$ convergence.

general case: may not always be the case where $a_n \geq 0$, so $\langle S_N \rangle$ may not be monotone $\therefore$ not enough to show that $\langle S_N \rangle$ is bounded.

for instance, $\sum_n (-1)^{n+1} = 1-1+1-1+\cdots$ has partial sums $1, 0, 1, 0, 1, 0 \ldots$ (bounded but divergent).

recall: $\sum_n a_n$ is absolutely convergent if $\sum_n |a_n| < \infty$ (converges).

$\boxed{\text{Definition}}$ if $\sum_n a_n$ converges but not absolutely, we say that it is conditionally convergent.

$\boxed{\text{Ex}}$ (oscillating harmonic series) $\sum_n \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. this converges conditionally.

Proof – $\sum_n \left|\frac{(-1)^{n+1}}{n}\right| = \sum_n \frac{1}{n}$, which diverges $\Rightarrow$ not absolutely convergent.

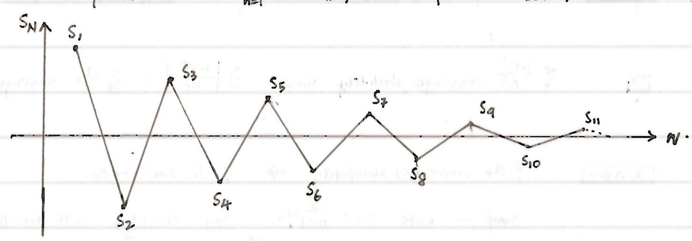
$\sum_n \frac{(-1)^{n+1}}{n}$ converges due to the alternating series test.

ALTERNATING SERIES TEST.

$\boxed{\text{Theorem}}$ Assume $\langle a_n \rangle$ is a monotone decreasing sequence with $a_n \geq 0$ and $a_n \to 0$.

Then $\sum_n (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots\cdots$ converges.

Sketch of proof – consider the partial sums $S_N = \sum_{n=1}^{N} (-1)^{n+1} a_n$, and subsequences $\langle S_{2N} \rangle$, $\langle S_{2N+1} \rangle$.



$\langle S_{2N+1} \rangle$ is decreasing but bounded below by S_2; $\langle S_{2N} \rangle$ is increasing but bounded above by S_1.

$\Rightarrow$ both converge, and $S_{2N+1} \to L$ and $S_{2N} \to l$.

note that $a_n \to 0$ as $n \to \infty$, so $l = L$.

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convergence tests for $\sum_n a_n$

(1) if $a_n \not\to 0$, $\sum_n a_n$ diverges.

(2) if $a_n \geq 0\ \forall n$

(a) comparison test. (a') limit comparison test.

standard examples:
(i) geometric series $\sum_n x^n$ (ii) telescoping series. (iii) Riemann-zeta function. [though we cannot compute its limit yet].

(b) ratio test

(c) root test. $\leftarrow$ integral test!

(3) signs of terms alternate, e.g. $\sum_n (-1)^n a_n$ for $a_n \geq 0$: alternating series test.

(4) if none of the above apply, we use property of test (2), and test for absolute convergence.

(5) and should all that fail... give up.

Recall: for a geometric series, $\sum_{n=0}^{\infty} x^n$ converges $\Leftrightarrow$ $|x|<1$.

idea – consider $\sum_n a_n$ that behave like a geometric series as $n\to\infty$, then

(i) if $a_n = x^n$, then $x = \frac{a_{n+1}}{a_n}$ (ii) $x = \sqrt[n]{a_n}$.

$\boxed{\text{Theorem}}$ RATIO TEST:

if $a_n > 0$ $\forall n$, suppose $\frac{a_{n+1}}{a_n} \to l \in \mathbb{R}$ as $n\to\infty$; then

- if $l<1$, the series converges
- if $l>1$, the series diverges (i.e. $\sum_n a_n = \infty$).

Proof – since $\frac{a_{n+1}}{a_n} \to l$, for any $\varepsilon>0$ $\exists N>0$ s.t. $n>N \Rightarrow \left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon \Leftrightarrow l-\varepsilon < \frac{a_{n+1}}{a_n} < l+\varepsilon \Leftrightarrow$

$a_n(l-\varepsilon) < a_{n+1} < (l+\varepsilon)a_n$

then if $N'>N$ and $k\in\mathbb{N}$, then $\boxed{a_{N'}(l-\varepsilon)^k < a_{N'+k} < (l+\varepsilon)^k a_{N'}}$

if $l<1$, chose $\varepsilon>0$ small enough s.t. $l+\varepsilon<1$, then

$$\sum_{n=N'}^{\infty} a_n = \sum_{k=0}^{\infty} a_{N'+k} < \sum_{k=0}^{\infty} (l+\varepsilon)^k a_{N'} = a_{N'} \frac{1}{1-(l+\varepsilon)} < \infty.$$

$\therefore$ by comparison with a convergent geometric series $\sum_{n=N'}^{\infty} a_n < \infty \Rightarrow \sum_n a_n$ converges$_{//}$ q.e.d.

now suppose $l>1$, choose $\varepsilon>0$ small enough s.t. $l-\varepsilon>1$, then $\sum_{n=N'}^{\infty} a_n > \sum_{k=0}^{\infty} (l-\varepsilon)^k a_{N'} = +\infty$.

$\boxed{\text{Theorem}}$ ROOT TEST:

if $a_n>0$ $\forall n$; suppose $\sqrt[n]{a_n} \to l \in \mathbb{R}$; then

- $l<1 \Rightarrow \sum_n a_n$ converges
- $l>1 \Rightarrow \sum_n a_n$ diverges.

Proof – compare with a geometric series for large n (as in ratio test).

Applications — Power Series.

$\boxed{\text{Ex}}$ $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ (remark, to be proved later: this yields e^x).

Proof of convergence – write series $\sum_{n=0}^{\infty} a_n$ with $a_n = \frac{|x|^n}{n!}$

then defining $b_n = \frac{|x|^n}{n!}$, we try the ratio test: $\frac{b_{n+1}}{b_n} = |x| \cdot \frac{n!}{(n+1)!} = \frac{|x|}{n+1} \to 0 < 1$ as $n\to\infty$.

$\therefore$ ratio test $\Rightarrow \sum_n \left|\frac{x^n}{n!}\right| < \infty \Rightarrow$ absolute convergence $\Rightarrow$ convergence$_{//}$ q.e.d.

$\boxed{\text{Ex}}$ Show $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ can converge (remark: this is $\arctan x$)

When does it converge absolutely? note that we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1}$ converges absolutely $\Leftrightarrow \sum_{n=1}^{\infty} \frac{|x|^{2n-1}}{2n-1} < \infty$.

write $|a_n| = \frac{|x|^{2n-1}}{2n-1}$, $\frac{|a_{n+1}|}{|a_n|} = |x|^2 \frac{2n-1}{2n+1} \to |x|^2$

by the ratio test, $|x|<1 \Rightarrow$ series converges absolutely$_{//}$ and $|x|>1 \Rightarrow$ series diverges.

for $x=\pm1$, the series becomes $\pm(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$, converges by the alternating series test.

it will converge absolutely $\Leftrightarrow 1 + \frac{1}{3} + \frac{1}{5} + \cdots$ converges; but it diverges: $1 + \frac{1}{3} + \frac{1}{5} > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3}) \Rightarrow$ divergence.

thus, for $x=\pm1$, the series converges conditionally$_{//}$.

REARRANGEMENTS.

we have seen that (i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges conditionally, and (ii) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ converges absolutely.

consider subsequences of the terms $\langle a_n \rangle$ defined as $\langle p_n \rangle$ = all terms in $\langle a_n \rangle$ s.t. $a_n \geq 0$; $\langle m_n \rangle$ = all terms in $\langle a_n \rangle$ s.t. $a_n < 0$.

then for (i), $\langle p_n \rangle = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$, $\langle m_n \rangle = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots$, then $\sum_n p_n = +\infty$, $\sum_n m_n = -\infty$.

and for (ii), $\langle p_n \rangle = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$, $\langle m_n \rangle = -\frac{1}{4} - \frac{1}{16} - \frac{1}{36} - \cdots$, then $\sum_n p_n < \sum_n \frac{1}{n^2} < \infty$, and $-\sum_n m_n < \sum_n \frac{1}{n^2} < \infty \Rightarrow$ both converge

back to (i), we know that it converges, and define $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = S$. $\Rightarrow (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots = S$.

also, $\frac{1}{2} \quad - \frac{1}{4} \quad + \frac{1}{6} \quad + \cdots = \frac{1}{2}S$.

adding them: $(1 - \frac{1}{2} + \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6} + \frac{1}{6}) + (\frac{1}{7} - \frac{1}{8} - \frac{1}{8}) + \cdots = \frac{3}{2}S$

and hence, (iii) $1 + (\frac{1}{3} - \frac{1}{2}) + \frac{1}{5} + (\frac{1}{7} - \frac{1}{4}) + \cdots = \frac{3}{2}S$, then we notice that this is series (i), but only with its terms rearranged. (added in a different order)

(i): $p_1 + m_1 + p_2 + m_2 + p_3 + m_3 + \cdots = S$. (iii): $(p_1 + p_2) - m_1 + (p_3 + p_4) - m_2 + \cdots = \frac{3}{2}S$. $\Rightarrow$ order sometimes matters!

$\boxed{\text{Theorem}}$ Suppose $\sum_n a_n$ converges conditionally. then, $\forall\, l \in \mathbb{R}$, the terms of $\sum_n a_n$ can be red[illegible] s.t. they sum to l.

i.e. different order $\Rightarrow$ different series ($\because$ partial sums change).

sketch of proof — we assume $\sum_n a_n$ converges but $\sum_n |a_n| = +\infty$.

lemma: $\sum_n a_n$ is conditionally convergent $\Rightarrow$ $\sum_n p_n = +\infty$ and $\sum_n m_n = -\infty$.

Proof – consider $S_N = \sum_{n=1}^{N} a_n$ and $A_n = \sum_{n=1}^{N} |a_n|$; then as $N \to \infty$, S_N converges but A_n diverges.

$S_N = \sum_{n=1}^{J} p_n - \sum_{n=1}^{K} |m_n|$ for some $J, K \in \mathbb{N}$; whereas $A_N = \sum_{n=1}^{J} p_n + \sum_{n=1}^{K} |m_n|$.
$= P - M$ $\qquad\qquad = P + M$.

so as $N \to \infty$, $P - M$ is bounded but $P + M$ is unbounded. $\Rightarrow$ P, M are not both bounded.

but for $P - M$ is bounded, then both P and M must both be unbounded.

P, M are partial sums of $\sum_n p_n$ and $\sum_n m_n$ respectively; unbounded $\Rightarrow$ divergence. // q.e.d.

(remark: the lemma would not hold if $\sum_n a_n$ is absolutely convergent).

A recipe for a reordering of $\sum_n a_n$ convergent to $l \in \mathbb{R}$:

steps

(1) add positive terms until sum $\geq l$ (can do this because $\sum_n p_n$ diverges – lemma)

(2) add negative terms until sum $\leq l$. (can do this because $\sum_n m_n$ diverges to $-\infty$).

(3) continue iterating steps (1) and (2) on repeat $\Rightarrow$ we will obtain partial sums oscillating around l.

as terms are used up, smaller terms remain to be applied, as $\sum_n a_n$ converges $\Rightarrow$ $a_n \to 0$ as $n \to \infty$ // q.e.d.

$\boxed{\text{Theorem}}$ if $\sum_n a_n$ converges absolutely, then all reorderings of the series have the same sum.

e.g. $\sum_n \frac{(-1)^n}{n^2}$.

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Ratio test: we saw that $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1}$ this converges absolutely iff $\sum_n |a_n| < \infty$.

$\frac{|a_{n+1}|}{|a_n|} = |x|^2 \frac{2n-1}{2n+1} \to |x|^2$ as $n \to \infty$.

$\therefore$ if $|x| < 1$, then $\sum_n |a_n| < \infty \Rightarrow \sum_n a_n$ converges absolutely.

if $|x| > 1$, then $\sum_n |a_n| = \infty$; but could $\sum_n a_n$ converge conditionally? no...

recall: if $\frac{|a_{n+1}|}{|a_n|} \to l > 1$ $\Rightarrow$ we can pick $\varepsilon > 0$ s.t. $l - \varepsilon > 1$ and find $N > 0$ s.t. $n > N \Rightarrow \left| \left|\frac{a_{n+1}}{a_n}\right| - l \right| < \varepsilon$

i.e. $l - \varepsilon < \frac{|a_{n+1}|}{|a_n|} < l + \varepsilon$; and thus $\frac{|a_{n+1}|}{|a_n|} > l - \varepsilon > 1$ $\Rightarrow$ $|a_{n+1}| > (l - \varepsilon)|a_n|$.

now if $N' > N$; $k \in \mathbb{N}$ we have $|a_{N'+k}| > (l-\varepsilon)|a_{N'+k-1}| > \dots > (l-\varepsilon)^k |a_{N'}|$

$\therefore$ as $k \to +\infty$, $|a_{N'+k}| \to \infty$, $\therefore |a_n| \to \infty$ (series does not converge abs.) $\Rightarrow$ $a_n \not\to 0$ (series diverges).

$\boxed{\text{Theorem}}$ RATIO TEST (part 2):

Given any series $\sum_n a_n$ (no need to assume $a_n > 0$), if $\frac{|a_{n+1}|}{|a_n|} \to l$, then

(i) if $l < 1 \Rightarrow \sum_n a_n$ converges absolutely.

(ii) if $l > 1 \Rightarrow \sum_n a_n$ diverges.

FUNCTIONS.

Given two sets $A, B \subset \mathbb{R}$, a <u>function</u> from A to B (notation: "$f: A \to B$") assigns to each $x \in A$ a unique element $f(x) \in B$.

then A is the "<u>domain</u> of the function" and B is the "<u>target</u> of F" (not the <u>range</u>! range $\leq$ target: the range of F is the set $F(A) = \{f(x) \in B \mid x \in A\} \subset B$.

recall from MATH1201 –

$f: A \to B$ is <u>injective</u> if whenever $x, y \in A$ s.t. $x \neq y$, $f(x) \neq f(y)$

$f: A \to B$ is <u>surjective</u> if $f(A) = B$.

For a subset $C \subset A$, the <u>image</u> of C under F is $f(C) = \{f(x) \in B \mid x \in C\}$.

$f: A \to B$ is <u>bijective</u> if it is both injective and surjective.

composition: suppose $g: A \to B$, $f: C \to D$ s.t. $g(A) \underbrace{= B}_{\text{surjective!}} \subset C$, then we can define $f \circ g : A \to D$, $x \overset{\text{"maps to"}}{\longmapsto} f(g(x))$.

$\boxed{\text{Ex}}$ Let $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$

then domain is $\mathbb{R}$, range is $\mathbb{R}^+ \cup \{0\}$; <u>not injective</u> since $\forall x \neq 0$, $f(x) = f(-x)$ but $x \neq -x$.

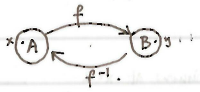
<u>not surjective</u> since $\mathbb{R}^+ \cup \{0\} = f(\mathbb{R}) \neq \mathbb{R}$.

$\boxed{\text{Ex}}$ $f: (-\infty, 0] \to [0, \infty) : x \mapsto x^2$

this is both <u>injective</u> and <u>surjective</u>.

Proposition — Any bijective function $f: A \mapsto B$ has a unique inverse $f^{-1}: B \to A$ s.t.

$f^{-1} \circ f : A \to A : x \mapsto x$ and $f \circ f^{-1} : B \mapsto B : x \mapsto x$ (or $y \mapsto y$).



surjective: y is in image of $f(A)$
injective: for each y, x is unique.

$\boxed{\text{Ex}}$ $f: (-\infty, 0] \to [0, \infty) : x \mapsto x^2$, then $f^{-1}: [0, \infty) \to (-\infty, 0] : y \mapsto -\sqrt{y}$ ← due to domain of f!

$\boxed{\text{Definition}}$ Assume $A, B \subset \mathbb{R}$. Then a function $f: A \mapsto B$ <u>is bounded above/below</u> if <u>its range is bounded above/below</u>.

$f: A \mapsto B$ is <u>(strictly) monotone increasing</u> if $x, y \in A$, $x > y \Rightarrow f(x) \geq f(y)$ (strictly: $f(x) > f(y)$)

<u>(strictly) monotone decreasing</u> if $x, y \in A$, $x > y \Rightarrow f(x) \leq f(y)$ (strictly: $f(x) < f(y)$).

$\boxed{\text{Ex}}$ We know that a function $f: \mathbb{N} \to \mathbb{R}$ is a sequence: $\langle f(n) \rangle$.

if $g: \mathbb{N} \to \mathbb{N}$ is <u>strictly increasing</u>, then $f \circ g: \mathbb{N} \to \mathbb{R}$ is a subsequence $\langle f(g(n)) \rangle$.

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ULT.

LIMITS AND CONTINUITY.

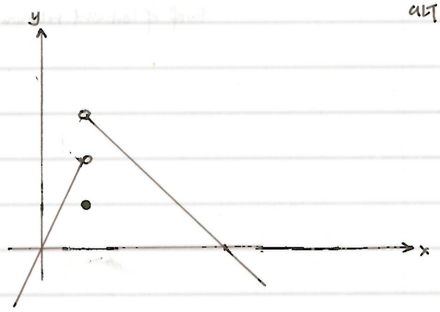
$\boxed{\text{Ex}}$ $f(x) = \begin{cases} 2x & \text{if } x < 1, \\ 1 & \text{if } x = 1, \\ 4-x & \text{if } x > 1. \end{cases}$

For this piece-wise function,

$f(1) = 1$,

$\lim_{x \to 1^-} f(x) = 2$ or $\lim_{x \nearrow 1} f(x) = 2$

$\lim_{x \to 1^+} f(x) = 3$ or $\lim_{x \searrow 1} f(x) = 3$.



$\boxed{\text{Definition}}$ Given a function $f: (a,b) \mapsto \mathbb{R}$ and a number $l \in \mathbb{R}$, we say $\lim_{x \to a^+} f(x) = l$ if

given any $\varepsilon > 0$, $\exists \delta > 0$ s.t. $a < x < a + \delta \Rightarrow |f(x) - l| < \varepsilon$. (or equivalently, $l - \varepsilon < f(x) < l + \varepsilon$.)

(equivalently: if $x \in (a, a+\delta) \Rightarrow f(x) \in (l - \varepsilon, l + \varepsilon)$)

We say that $\lim_{x \to b^-} f(x) = l$ if

given any $\varepsilon > 0$, $\exists \delta > 0$ s.t. $b - \delta < x < b \Rightarrow |f(x) - l| < \varepsilon$

(equivalently: if $x \in (b - \delta, b) \Rightarrow f(x) \in (l - \varepsilon, l + \varepsilon)$).

If f is defined on (a,b) except possibly at some point $c \in (a,b)$, then f is defined at least on the intervals (a,c) and (c,b).

we say that $\lim_{x \to c} f(x) = l$ if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = l$. Does not depend on value of $f(c)$, even if it exists).

(equivalently, given $\varepsilon > 0$, $\exists \delta$ s.t. $x \in (c - \delta, c + \delta) \Rightarrow f(x) \in (l - \varepsilon, l + \varepsilon)$; or

if $|x - c| < \delta$ and $x \neq c$ $\Rightarrow |f(x) - l| < \varepsilon$).

For instance, using an applet, we propose that $\lim_{x \to 0} (1 - x^2 \cos(\frac{1}{x})) = 1$.

Returning to the earlier example,

Ex (cont'd) We claim $\lim_{x \to 1^-} f(x) = 2$.

Proof — Given $\varepsilon > 0$, we must show that $\exists \delta > 0$ s.t. $x \in (1 - \delta, 1) \Rightarrow f(x) \in (2 - \varepsilon, 2 + \varepsilon) \Rightarrow 2 - \varepsilon < f(x) < 2 + \varepsilon$ (for $x < 1$) $\Rightarrow$

$2 - \varepsilon < 2x < 2 + \varepsilon \Rightarrow 1 - \frac{\varepsilon}{2} < x < 1 + \frac{\varepsilon}{2}$ $\therefore$ given $\varepsilon > 0$, it suffices to choose $\delta = \frac{\varepsilon}{2}$, and hence

$x \in (1 - \delta, 1) \Rightarrow x \in (1 - \frac{\varepsilon}{2}, 1) \Rightarrow f(x) \in (2 - \varepsilon, 2) \in (2 - \varepsilon, 2 + \varepsilon)$, q.e.d.

Proof for $\lim_{x\to 1^+} f(x)=3$.

Given $\varepsilon>0$, we must find $\delta>0$ s.t. $x\in(1,1+\delta) \Rightarrow f(x)\in(3-\varepsilon,3+\varepsilon)$.

now if $x>1$, then $f(x)=4-x$ so $3-\varepsilon<f(x)<3+\varepsilon \Rightarrow -1-\varepsilon<-x<-1+\varepsilon \Rightarrow 1+\varepsilon>x>1-\varepsilon$

assuming $x>1$, $1<x<1+\varepsilon$. set $\delta=\varepsilon$ // q.e.d.

(i.e. if $\delta=\varepsilon$, $x\in(1,1+\delta) \Rightarrow f(x)\in(3-\varepsilon,3+\varepsilon)$ //)

$\boxed{\text{Theorem}}$ $\lim_{x\to c} f(x)=l \iff$ for every sequence $\langle x_n\rangle$ with x_n in the domain of f $(\neq c)$ $\forall n$ and $x_n\to c$, we have $f(x_n)\to l$. (convergence of sequences)

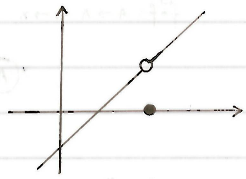
may not include c!

$\lim_{x\to c^-} f(x)=l \iff$ the above statement is true $\forall$ sequences $\langle x_n\rangle$ also satisfying $x_n>c$ $\forall n$.

$\lim_{x\to c^+} f(x)=l \iff$ the above statement is true $\forall$ sequences $\langle x_n\rangle$ also satisfying $x_n<c$ $\forall n$.

consider the function

$$f(x)=\begin{cases} x-1 & \text{if } x\neq 2 \\ 0 & \text{if } x=2 \end{cases} \Rightarrow f(x) \text{ is not continuous at } x=2.$$



$\boxed{\text{Definition}}$ A function f defined on (a,b) is continuous at a point $c\in(a,b)$ if $f(c)=\lim_{x\to c} f(x)$.

we also deduce that given $\varepsilon>0$, $\exists\delta>0$ s.t. $|x-c|<\delta$ and $x\neq c \Rightarrow |f(x)-l|<\varepsilon \iff$ for any seq. $\langle x_n\rangle$ s.t. $x_n\in$ domain of f; $x_n\neq c$ $\forall n$, $x_n\to c$ implies $f(x_n)\to l$.

Proof of forward relation — assume $f(x)=l$. Given a sequence $x_n\to c$ s.t. $x_n\neq c$, we need to show $f(x_n)\to l$.

that would mean that given $\varepsilon>0$, $\exists N>0$ s.t. $n>N \Rightarrow |f(x_n)-l|<\varepsilon$.

since $\lim_{x\to c} f(x)=l$, we know $\exists\delta>0$ s.t. $\forall x\in(c-\delta,c+\delta)$ with $x\neq c$; $|f(x)-l|<\varepsilon$.

$\therefore$ our desired statement is true whenever $|x_n-c|<\delta$.

now since $x_n\to c$, $\exists N'>0$ s.t. $n>N' \Rightarrow |x_n-c|<\delta$

so $n>N' \Rightarrow |x_n-c|<\delta \Rightarrow |f(x_n)-l|<\varepsilon$, $\therefore f(x_n)\to l$ // q.e.d.

Proof of backward relation — we prove the contrapositive, i.e. NTP

if $\lim_{x\to c} f(x)\neq l$, then not every sequence $x_n\to c$ with $x_n\neq c$ yields $f(x_n)\to l$.

assuming $\lim_{x\to c} f(x)=l$, then for some $\varepsilon>0$, every $\delta>0$ has the property: $|x-c|<\delta$ and $x\neq c \not\Rightarrow |f(x)-l|<\varepsilon$.

in other words: $\exists x\in(c-\delta,c+\delta)$ with $x\neq c$ and $|f(x)-l|\geq\varepsilon$.

in particular, $\exists$ an $\varepsilon>0$ s.t. this is true for arbitrarily small $\delta>0$.

$\therefore$ we can find a sequence $\langle x_n\rangle$ s.t. by setting $\delta=\frac{1}{n}$, we may assume $x_n\in(c-\frac{1}{n},c+\frac{1}{n})$, $x_n\neq c$.

yet $|f(x_n)-l|\geq\varepsilon$. $\therefore$ our sequence $x_n\to c$, but $f(x_n)\not\to l$ // q.e.d.

$\boxed{\text{Corollary}}$ A function $f:(a,b)\to\mathbb{R}$ is continuous at $c\in(a,b) \iff$ for every sequence $x_n\in(a,b)$ with $x_n\to c$, we have $f(x_n)\to \underline{f(c)}$.

actual value of fn!

$\boxed{\text{Corollary}}$ "combination theorem" for limits of functions —

suppose $\lim_{x\to c} f(x)=l$ and $\lim_{x\to c} g(x)=m$. Then

(i) $\lim_{x\to c}[f(x)+g(x)]=l+m$

(ii) For any constant $a\in\mathbb{R}$, $\lim_{x\to c}[a f(x)]=al$.

(iii) $\lim_{x\to c}[f(x)g(x)]=ml$.

(iv) if $m\neq 0$, then $\lim_{x\to c}\frac{1}{g(x)}=\frac{1}{m}$.

Proof of (i) — $\lim_{x\to c} f(x)=l$ and $\lim_{x\to c} g(x)=m \Rightarrow \forall$ sequences $x_n\to c$ with $x_n\neq c$, $f(x_n)\to l$ and $g(x_n)\to m$.

$\therefore$ by the combination theorem for sequences, $f(x_n)+g(x_n)\to m+l$.

since this is true for all sequences $x_n\to c$ with $x_n\neq c$, we conclude that $\lim_{x\to c}[f(x)+g(x)]=l+m$ // q.e.d.

Proofs of (ii), (iii), (iv) are analogous.

lemma — The function $f(x)=x$ is continuous everywhere.

Proof — we need to prove that $\forall c\in\mathbb{R}$, $c=\lim_{x\to c} f(x)=\lim_{x\to c} x$.

lemma — every constant function is continuous.

corollary: by the combination theorem,

every polynomial function is continuous everywhere;

and so is every rational function $f(x)=\frac{P(x)}{Q(x)}$ where P,Q are polynomials, as long as $Q(x)\neq 0$.

Proposition — Given functions f and g which are continuous at a point c, the functions $f+g$ and fg (product) are also continuous at c, and if $g(c) \neq 0$, so is f/g.

Proof — assume f, g are continuous at c, is $f+g$ continuous at c?

$$f(c)+g(c) \overset{?}{=} \lim_{x\to c}[f(x)+g(x)] \underset{\text{combination theorem}}{=} \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \underset{\text{continuity}}{=} f(c)+g(c) \;// \text{ q.e.d.}$$

1 December 2011
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Theorem Assume f is a function defined on (a,b), except possibly at some point $c\in(a,b)$.
Then $\lim_{x\to c} f(x) = l \iff$ for all sequences $\langle x_n\rangle$ with $x_n \in (a,b)$, $x_n\neq c\ \forall n$ but $x_n\to c$, $f(x_n)\to l$.

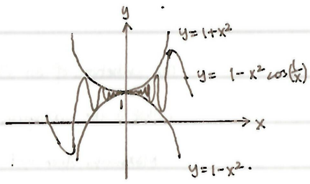
Definition If f is defined on some interval (a,∞), we say $\lim_{x\to\infty} f(x)=l$ if given $\varepsilon>0$, we can find $H\in\mathbb{R}$ $(H>a)$ s.t. $x>H \Rightarrow |f(x)-l|<\varepsilon$.
If f is defined on some interval $(-\infty, b)$, we say $\lim_{x\to-\infty} f(x)=l$ if given $\varepsilon>0$, we can find $H\in\mathbb{R}$ $(H<b)$ s.t. $x<H \Rightarrow |f(x)-l|<\varepsilon$.

Ex Prove that $\lim_{x\to\infty}\frac{1}{x}=0$.
Given $\varepsilon>0$, we want $H\in\mathbb{R}$ s.t. $x>H \Rightarrow \left|\frac{1}{x}-0\right|<\varepsilon \iff \frac{1}{x}<\varepsilon \iff x>\frac{1}{\varepsilon}$. We take $H=\frac{1}{\varepsilon}$ // q.e.d.

Theorem Assume f is a function defined on (a,∞). Then $\lim_{x\to+\infty} f(x)=l \iff \forall$ sequences $\langle x_n\rangle$ with $x_n\in(a,\infty)\ \forall n$ and $x_n\to+\infty$, $f(x_n)\to l$.
Proof of forward relation — Assume $\lim_{x\to+\infty} f(x)=l$ and $\langle x_n\rangle$ is a sequence with $x_n\to+\infty$.
then given $\varepsilon>0$, $\exists H\in\mathbb{R}$ s.t. $x>H \Rightarrow |f(x)-l|<\varepsilon$.
also, $\exists N>0$ s.t. $n>N \Rightarrow x_n>H$.
$\therefore \forall n>N$, $x_n>H \Rightarrow |f(x_n)-l|<\varepsilon$; therefore $f(x_n)\to l$ // q.e.d.

Corollary SANDWICH THEOREM for functions.
Suppose f,g,h are functions defined on (a,b), except possibly at $c\in(a,b)$; and $f(x)\le g(x)\le h(x)\ \forall x$,
and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = l$.
then $\lim_{x\to c} g(x) = l$. (note: can also apply where $c=\pm\infty$).
Proof — It suffices to show that $\forall$ sequences $x_n\neq c$ but $x_n\to c$, we have $g(x_n)\to l$.
notice that $\forall n$, $f(x_n)\le g(x_n)\le h(x_n)\ \forall n$ and $f(x_n)\to l$ and $h(x_n)\to l$.
we then apply sandwich theorem for sequences // q.e.d.

Ex $f(x)=1-x^2\cos\left(\frac{1}{x}\right)$ for $x\neq0$. Prove the claim that $\lim_{x\to0} f(x)=1$.
we know that $-1\le\cos\frac{1}{x}\le1 \Rightarrow 1-x^2\le f(x)\le 1+x^2$.
since polynomials are continuous, $1-x^2$ and $1+x^2$ are continuous, and
$\lim_{x\to0} f(x) = \lim_{x\to0}(1-x^2) = \lim_{x\to0}(1+x^2) = 1$ // q.e.d.



Ex Find $\lim_{x\to\infty}(\sin x)$; and prove it.
claim: limit does not exist.
Proof — by contradiction. For all sequences $x_n\to+\infty$, limit exists $\Rightarrow \sin x_n$ must converge to $l\in\mathbb{R}$.
Consider $x_n=\frac{\pi}{2}n$, where $x_n\to+\infty$, $\langle\sin x_n\rangle = 1,0,-1,0,1,0,-1,0,\ldots$ has subsequences converging to different limits $\Rightarrow$
$\sin x_n$ diverges $\Rightarrow$ contradiction! so limit does not exist // q.e.d.

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If $\lim_{x\to c} g(x)=l$ and $\lim_{x\to l} f(x)=m$, then $\lim_{x\to c}[f\circ g(x)] \overset{?}{=} m$. Not necessarily.
Counterexample: $g(x)=2$, $f(x)=\begin{cases}1 & \text{if } x=2\\ 0 & \text{if } x\neq2\end{cases}$
then $\lim_{x\to0} g(x)=2$; $\lim_{x\to2} f(x)=0\neq f(2)=1$. $\therefore \lim_{x\to0}[f\circ g(x)]=1\neq0$. Why?
The discontinuity here results in the statement being untrue.

In general, for the following, assume that $\lim_{x\to c} g(x) = l$ and $\lim_{x\to l} f(x) = m$.

Proposition — if f is continuous at l, then $\lim_{x\to c} [f\circ g(x)] = m$.

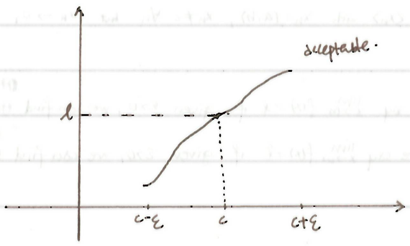
Proof — It suffices to show that for any sequence $\langle x_n \rangle$ s.t. $x_n \neq c$, $x_n \to c$, then $f\circ g(x_n) \to m$.

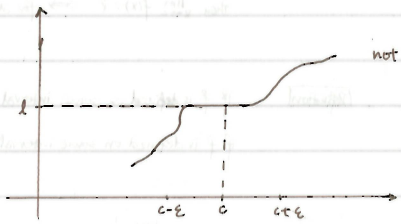
we have $g(x_n) \to l$, then by continuity of f at l, we have $f(l) = f(\lim_{x\to c} g(x_n)) = \lim_{x\to c} f(g(x_n))$

$= \lim_{x\to l} f(x) = m$, q.e.d.

Corollary — If g is continuous at c, and f is continuous at $g(c)$, then $f\circ g$ is also continuous at $x=c$.

Proposition — if $\exists\, \varepsilon>0$ s.t. $g(x) \neq l$ on interval $x\in(c-\varepsilon, c+\varepsilon)$; except possibly at c, then $\lim_{x\to c} [f\circ g(x)] = m$.





Proof — assume $x_n \to c$ but $x_n \neq c$ $\forall n$. we know $g(x_n) \to l$; but $\exists N>0$ s.t. $\forall n > N$, $x_n \in (c-\varepsilon, c+\varepsilon)$ but $x_n \neq c$, $\therefore g(x_n) \neq l$.

since $\lim_{x\to l} f(x) = m$, $f(g(x_n)) \to m$ $\therefore \lim_{x\to c} [f\circ g(x)] = m$, q.e.d.

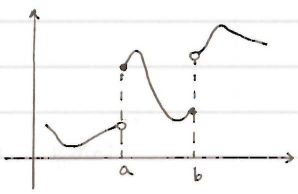
More on continuity:

Assume f has domain continuity (a,b) on $[a,b]$.

Definition f is <u>continuous on (a,b)</u> if it is continuous at x $\forall x\in(a,b)$.

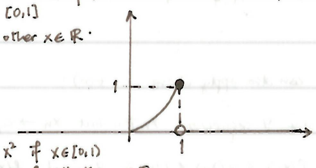
f is <u>continuous on $[a,b]$</u> if it is continuous on (a,b) and

$\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.



Ex $f(x) = \frac{1}{x-1}$ for $x\neq 1$. Then f is continuous on $[0,1)$, but not on $[0,1]$.

Ex $g(x) = \begin{cases} x^2 & \text{if } x\in[0,1] \\ 0 & \text{for all other } x\in\mathbb{R} \end{cases}$



$g(x)$ is continuous on $[0,1]$, not continuous on $\mathbb{R}$ as it is discontinuous (only "continuous from the left") at 1.

However, if $h(x) = \begin{cases} x^2 & \text{if } x\in[0,1) \\ 0 & \text{for all other } x\in\mathbb{R}, \end{cases}$

$h(x)$ is not continuous on $[0,1]$ since $h(1) = 0 \neq \lim_{x\to 1^-} h(x) = 1$.

The archetype of an *ill-posed problem*:

- What is the maximum value of the function $f(x) = \frac{1}{x}$ for $x\in(0,1]$?

  Maximum does not exist, since f is not bounded on $(0,1]$.

- For $f(x)$, given an interval, what will guarantee that (1) it is bounded, (2) it achieves a max and min?

Ex (cont'd) consider $h(x) = \begin{cases} x^2 & \text{if } x\in[0,1) \\ 0 & \text{for all other } x\in\mathbb{R} \end{cases}$ again.

$h(x)$ is bounded on $\mathbb{R}$.

$\therefore \exists \sup_{x\in[0,1]} h(x) = \sup\{h(x) \mid x\in[0,1]\} = 1$; but $\nexists \max_{x\in[0,1]} h(x)$ since $\nexists\, x\in[0,1]$ s.t. $h(x) = 1$.

Definition A <u>compact interval</u> is any interval in $\mathbb{R}$ that is closed and bounded.

i.e. any interval of the form $[a,b]$ for $a,b\in\mathbb{R}$, $a\le b$.

Theorem (i) Every continuous function on a compact interval is bounded.

(ii) Every continuous function on a compact interval achieves maximum and minimum values.

i.e. if $f: [a,b] \to \mathbb{R}$ is continuous, then $\exists\, x_-, x_+ \in [a,b]$ s.t.

$f(x_-)$ is a lower bound and $f(x_+)$ is an upper bound for the image of f for $x\in[a,b]$.

Proof of (i) – Recall that by the Bolzano-Weierstrass theorem states that every bounded sequence has a convergent subsequence.

Assume $f: [a,b] \to \mathbb{R}$ is continuous and arguing by contradiction, assume that it is unbounded.

$\therefore \nexists H>0$ s.t. $|f(x)| \leq H \quad \forall x \in [a,b]$.

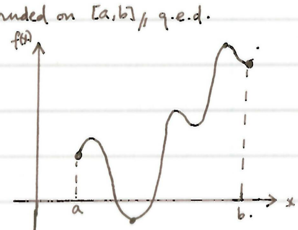
In particular, $\forall n \in \mathbb{N}$, $\exists$ some $x_n \in [a,b]$ s.t. $|f(x_n)| > n$. $\therefore |f(x_n)|$ is diverging to infinity.

but $x_n \in [a,b]$ is a bounded sequence, so by the Bolzano-Weierstrass theorem, $\exists$ a convergent sequence $\langle x_{j_n} \rangle$.

$x_{j_n} \to x_\infty \in \mathbb{R}$, and since $a \leq x_{j_n} \leq b$, then $a \leq x_\infty \leq b$. $\therefore$ x_∞ is in the compact interval on which f is continuous.

$\therefore$ since $x_{j_n} \to x_\infty$, $f(x_{j_n}) \to f(x_\infty)$. However, $\langle |f(x_{j_n})| \rangle$ is a subsequence of $\langle |f(x_n)| \rangle$, so $|f(x_n)| \to +\infty \Rightarrow |f(x_{j_n})| \to \infty$.

as such, $f(x_{j_n})$ is an unbounded but convergent sequence $\Rightarrow$ contradiction $\Rightarrow$ f is bounded on $[a,b]$ // q.e.d.



Proof of (ii) – from (i), we know that the continuous function in the interval $[a,b]$ is bounded.

$\Rightarrow \exists \sup_{x \in [a,b]} f(x) \in \mathbb{R}$. we know that

$\exists$ a sequence $y_n \in \{f(x) \mid x \in [a,b]\}$ s.t. $y_n \to \sup_{x\in[a,b]} f(x)$.

$\therefore$ $y_n = f(x_n)$ for some $x_n \in [a,b]$. By the Bolzano-Weierstrass theorem,

$\exists$ convergent subsequence $\langle x_{j_n} \rangle$ s.t. $x_{j_n} \to x_\infty \in [a,b]$. Now $\langle f(x_{j_n}) \rangle$ is a subsequence $\langle y_n \rangle$.

$\therefore$ $f(x_{j_n}) \to \sup_{x\in[a,b]} f(x)$. Also, since f is continuous, $f(x_{j_n}) \to f(x_\infty)$.

hence, we know that $f(x_\infty) = \sup_{x\in[a,b]} f(x)$ $\therefore$ $\max_{x\in[a,b]} f(x)$ exists and equals $f(x_\infty)$ // q.e.d.

for minimum, we apply the same proof, just replacing sup with inf // q.e.d.

Question: if $f: [a,b] \to \mathbb{R}$ is continuous, what does its image $f([a,b]) = \{f(x) \mid x \in [a,b]\}$ look like?

Answer 1 – we know at least that $f([a,b]) \subset [\min_{x\in[a,b]} f(x), \max_{x\in[a,b]} f(x)]$.

Question – Are those two intervals the same? Yes, because f is continuous and must pass through all intermediate points.

$\boxed{\text{Ex}}$ Take $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$

$f(-1) = -1$, $f(1) = 1$, but $0 \notin f([-1,1])$; since $f(x)$ is discontinuous.

$\boxed{\text{Theorem}}$ INTERMEDIATE VALUE THEOREM.

If $f: [a,b] \to \mathbb{R}$ is continuous, then any ℓ between $f(a)$ and $f(b)$ is in the image of f.

i.e. $\exists x \in [a,b]$ s.t. $f(x) = \ell$.

Corollary – the image of $[a,b]$ under f is another compact interval, namely $[\min_{x\in[a,b]} f(x), \max_{x\in[a,b]} f(x)]$.

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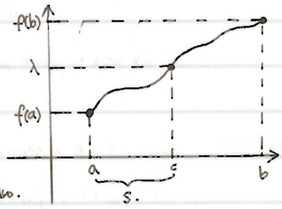

Proof – Let $S = \{x \in [a,b] \mid f(x) < \lambda\}$.

then the larger extreme of S is defined as $c = \sup S$.

claim 1 – $f(c) \leq \lambda$. $\exists$ a sequence $x_n \in S$ s.t. $x_n \to c$ (since S is bdd above)

then since $f(x_n) \to f(c)$, so $f(x_n) < \lambda \leq \lambda \Rightarrow f(c) = \lim f(x_n) \leq \lambda$ also.

claim 2 – $f(c) \geq \lambda$. by contradiction, suppose $f(c) < \lambda$; choose $\varepsilon > 0$, $f(c) + \varepsilon < \lambda$.

since f is continuous at $x = c$, $f(c) = \lim_{x \to c} f(x)$ $\therefore \exists \delta > 0$ s.t. $x \in (c, c+\delta) \Rightarrow |f(x) - f(c)| < \varepsilon$.

$\Rightarrow f(c) - \varepsilon < f(x) < f(c) + \varepsilon < \lambda$ $\therefore \exists x > c$ s.t. $f(x) < \lambda \Rightarrow x \in S$; thus c is not an upper bound of $S \Rightarrow$ contradiction

$\therefore$ $\lambda \leq f(c) \leq \lambda \Rightarrow f(c) = \ell$.

MORE ABOUT CONTINUITY.

$\boxed{\text{Definition}}$ Let $S \subset \mathbb{R}$ be any subset, $f: S \to \mathbb{R}$ is continuous if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $y \in S$ and $|y - x| < \delta \Rightarrow |f(x) - f(x)| < \varepsilon$.

$\boxed{\text{Ex}}$ If $S = [a,b]$ and $x = a$, this means $f(a) = \lim_{x \to a^+} f(x)$.

this gives us more definitions:

Definition (i) f is <u>continuous</u> on S if it is continuous at x $\forall x \in S$.

(ii) f is <u>uniformly continuous</u> on S if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $x, y \in S$ with $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

* note the difference: continuity – given x and ε, find δ ; uniform continuity – given ε, find δ, valid $\forall x$.

Application of uniform continuity – given $a>0$, what does a^x mean if $x \notin \mathbb{Q}$?

For: a^x where $x \in \mathbb{Q}$, $x = \frac{p}{q} \Rightarrow$ we define $a^x = \sqrt[q]{a^p}$

where $x \notin \mathbb{Q}$, though, we recall that every $x \in \mathbb{R}$ can be approximated by rationals.

i.e. $\exists$ sequence $x_n \in \mathbb{Q}$ s.t. $x_n \to x$. hence, we define $a^x = \lim_{x_n \to x} a^{x_n}$

problems – (I) we do not know if $\langle a^{x_n} \rangle$ converges.

see HW9 Problem – given that the function $f: \mathbb{Q} \to \mathbb{R}$ ($x \mapsto a^x$) is uniformly continuous on every bdd subset of $\mathbb{Q}$, then $\langle x_n \rangle$ is Cauchy $\Rightarrow \langle f(x_n) \rangle$ is also Cauchy $\Rightarrow$ converges.

(II) the value of a^x should depend only on x, but not on the sequence $\langle x_n \rangle$ which we have chosen.

(because $\exists$ infinitely many other sequences $y_n \in \mathbb{Q}$ s.t. $y_n \to x$).

so how do we know this? we use the theorem below:

Theorem if $f: S \to \mathbb{R}$ is uniformly continuous and $x_n, y_n \in S$ are sequences with $\lim x_n = \lim y_n$, then $\lim f(x_n) = \lim f(y_n)$.

Proof – given $x_n \to l$, $y_n \to l$, then $x_n - y_n \to l - l = 0$.

we want to show $f(x_n) - f(y_n) \to 0$. Given $\varepsilon > 0$,

(uniform continuity) $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

then $x_n - y_n \to 0$ means $\exists N > 0$ s.t. $n > N \Rightarrow |x_n - y_n| < \delta \Rightarrow \underbrace{|f(x_n) - f(y_n)| < \varepsilon}_{\to 0}$.

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CLT.

EXPONENTIAL FUNCTIONS.

For $a > 0$, the function $f: \mathbb{Q} \to (0, \infty): x \mapsto a^x$. is uniformly continuous on any bounded subset of $\mathbb{Q}$.

$\Rightarrow$ one can also define a^x for $x \notin \mathbb{Q}$ such that for any $x_n \in \mathbb{Q}$, $x_n \to x$, then $a^{x_n} \to a^x$.

we have shown that (i) $\langle a^{x_n} \rangle$ is Cauchy, and

(ii) $\lim a^{x_n}$ is dependent only on x, not on the sequence $\langle x_n \rangle$.

we still need to prove <u>uniform continuity</u>.

Theorem Assume $S \subset \mathbb{R}$ is either $\mathbb{Q}$ or $\mathbb{R}$, and $f: S \to (0, \infty)$ is a function such that

(i) $f(0) = 1$, $f(x) > 1$ $\forall x > 0$ and $f(x) < 1$ $\forall x < 0$.

(ii) $f(x+y) = f(x) \cdot f(y)$ $\forall x, y \in S$.

Then f is uniformly continuous on every bounded subset of S.

Ex. if $a > 1$, $f: \mathbb{Q} \to (0, \infty): x \mapsto a^x$ satisfies this assumption.

if $0 < a < 1$, then $f: \mathbb{Q} \to (0, \infty): x \mapsto a^{-x} = (\frac{1}{a})^x$ also does.

$\therefore$ in either case, this $\Rightarrow a^x$ is uniformly continuous on bounded subsets.

Corollary – we can extend the function $f(x) = a^x$ s.t. its domain includes all $x \in \mathbb{R}$, and if $\boxed{x_n \in \mathbb{Q}}$, $x_n \to x$, then $a^{x_n} \to a^x$.

not very general yet!

Now for any $x, y \in \mathbb{R}$, we claim: $a^{x+y} = a^x a^y$

Proof – $\exists$ sequences $\overset{\mathbb{Q}}{x_n} \to x$, $\overset{\mathbb{Q}}{y_n} \to y$; then by definition, $a^{x_n} \to a^x$ and $a^{y_n} \to a^y$.

then $x_n + y_n \in \mathbb{Q} \xrightarrow[\text{thm}]{\text{combination}} x+y$. $\Rightarrow a^{x_n + y_n} \to a^{x+y}$ $\therefore a^{x_n} \cdot a^{y_n} \xrightarrow{\text{combin thm}} a^{x+y} \Rightarrow a^x \cdot a^y = a^{x+y}$ // qed

1101-024.

Similarly, we can show that $a^x > 1 \ \forall x>0$, $a^x<1 \ \forall x<0$. $\therefore f: \mathbb{R} \to (0,\infty): x \mapsto a^x$ satisfies the conditions of theorem $\Rightarrow$ continuous and also uniformly continuous on all bounded subsets.

Remark: It is true that for any continuous function $f: [a,b] \to \mathbb{R}$ (compact interval), then f is also uniformly continuous on $[a,b]$.

Proof – given $\varepsilon>0$, $\exists \delta>0 \Rightarrow \delta$ is a function of x, $\delta(x)>0$ for $[a,b]$.

$\delta(x)$ is continuous, and bounded away from 0……

Proof (of theorem) – we assume $S = \mathbb{Q}$ or $\mathbb{R}$, $f: S \mapsto (0,\infty)$ satisfies

$f(x+y) = f(x)f(y)$ and $f(0)=1$, $f(x)>1 \ \forall x>0$, $f(x)<1 \ \forall x<0$.

step 1: if $f(1)=a$, then $\forall x \in \mathbb{Q}$, $f(x)=a^x$.

Proof – (i) $n \in \mathbb{N} \Rightarrow f(n) = f(1+\cdots+1) = f(1)\cdots f(1) = [f(1)]^n = a^n$.

(ii) $f(0)=1=a^0$. by assumption, for $n\in\mathbb{N}$, $\overset{1=f(0)}{=} f(n-n) = f(n)f(-n) \Rightarrow f(-n) = \frac{1}{f(n)} = \frac{1}{a^n} \Rightarrow f(-n) = a^{-n}$.

$\therefore f(n) = a^n \ \forall n \in \mathbb{Z}$.

(iii) $a = f(1) = f(\frac{n}{n}) = f(\frac{1}{n}+\cdots+\frac{1}{n}) = [f(\frac{1}{n})]^n \Rightarrow f(\frac{1}{n}) = \sqrt[n]{a} = a^{\frac{1}{n}}$.

(iv) for $\frac{p}{q}\in\mathbb{Q}$, $f(\frac{p}{q}) = f(\frac{1}{q}+\cdots+\frac{1}{q}) = [f(\frac{1}{q})]^p = (\sqrt[q]{a})^p = a^{\frac{p}{q}}$.

step 2: f is strictly increasing i.e. $x>y \Rightarrow f(x)>f(y)$.

Proof – $x>y \Leftrightarrow x-y>0 \Rightarrow f(x-y) = f(x)f(-y) = \frac{f(x)}{f(y)} > 1$.

by assumption, $f(x)>f(y)$.

step 3: claim f is continuous at 0.

Proof – We need to ~~show~~ $f(0)=1=\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x)$.

(i) For $\lim_{x\to0^+} f(x)$, it suffices to show that $\forall$ sequences $x_n \in S$, with $x_n>0$ and $x_n\to0$, we have $f(x_n)\to 1$.

$\exists$ a sequence $k_n\in\mathbb{N}$ with $k_n\to+\infty$ s.t. $0<x_n<\frac{1}{k_n}$. $\Rightarrow 1<f(x_n)<f(\frac{1}{k_n}) = a^{\frac{1}{k_n}}$

as $k_n\to\infty$, $\frac{1}{k_n}\to0$, $a^{\frac{1}{k_n}}\to1$; so $1<f(x_n)<1 \Rightarrow f(x_n)\to1$ by the sandwich theorem.

(ii) use the analogous argument to show that $\lim_{x\to0^-} f(x)=1$.

step 4: let $A \subseteq S$ be a bounded subset, so $\exists M>0$ s.t. $|x|\le M \ \forall x\in A$. $\Rightarrow -M\le x\le M \Rightarrow a^{-M}\le f(x)\le a^M$.

~~We know since f is continuous at~~ since $f(x)$ is monotone increasing, so WLOG, we can increase M to a rational number.

~~WLOG, assume $0 \in A$, to 0, given any $\varepsilon>0$, $\exists\delta>0$ s.t. $|x|<\delta \Rightarrow |f(x)-f(0)|<\varepsilon$.~~

$\therefore |f(x)|\le a^M$. Now, f is continuous at $0 \Rightarrow \forall\varepsilon>0$, $\exists\delta>0$ s.t. $|x|<\delta \Rightarrow |f(x)-f(0)|<\varepsilon$.

~~Then if $x,y\in A$, and $|x-y|<$~~ ~~; then $|f(x)-f(y)| = |f(x)[1-\frac{f(y)}{f(x)}]| = |f(x)[1-f(y-x)]| \le |f(x)||f(0)-f(y-x)|$.~~

Then $x,y\in A$ s.t. $|x-y|<\delta \Rightarrow |f(x)-f(y)| = |f(x)[1-\frac{f(y)}{f(x)}]| = |f(x)||1-f(y-x)| = |f(x)||f(0)-f(y-x)| < a^M\varepsilon$.

[Full Version] Given $\varepsilon>0$, $x,y\in A$ with $|x-y|<\frac{\varepsilon}{a^M} \Rightarrow |f(x)-f(y)|<\varepsilon$.

recall corollary: $f(x)=a^x$ defines a continuous function on $\mathbb{R}$ and is uniformly continuous on all bounded subsets.

$\boxed{\text{Definition}}$ $\forall x\in\mathbb{R}$, we define the function $\exp: \mathbb{R}\to\mathbb{R}$ by $\exp(x) = \sum_{n=0}^{\infty} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots$

exp converges absolutely for all x. (proof by ratio test).

$\boxed{\text{theorem}}$ The function exp satisfies the following properties.

(i) $\exp(x+y) = \exp(x)\exp(y)$

(ii) $\exp(x)>1 \ \forall x>0$, $\exp(x)\in(0,1) \ \forall x<0$ and $\exp(0)=1$.

corollary – by theorem earlier, $\exp:\mathbb{R}\to\mathbb{R}$ is continuous and always positive.

and if we define $e = \exp(1) = \sum_{n=0}^{\infty}\frac{1}{n!}$, then $\exp(x) = e^x \ \forall x$.

* further properties of e^x:

(1) e^x is strictly increasing (follows from proof of earlier theorem).

(2) $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.

Proof – $e^x = 1+x+\frac{x^2}{2!}+\cdots > 1+x$ for $x>0$. use limit comparison test.

as $x\to-\infty$, $-x\to\infty$ …… as above.

(3) e^x is bijective for $\mathbb{R}\to(0,\infty)$.

Proof – injective: $e^x = e^y \Rightarrow e^{x-y}=1 \Rightarrow x-y=0 \Rightarrow x=y$

surjective: by IVT.

(4) $\forall x\in\mathbb{R}$, $e^x \ge 1+x$.

Proof – $x>0$, obvious by power series. $x<-1$, $1+x\le0<e^x$. $-1\le x\le0$, ……

Proof – start with part (ii), $\exp(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots>1$ if $x>0$, and $\exp(x)=1$ if $x=0$.

now assuming part (i), $\exp(0)=1=\exp(x+(-x)) = \exp(x)\exp(-x) \Rightarrow \exp(-x) = \frac{1}{\exp(x)}$. $\therefore$ if $x>0$, then $\exp(x)>1 \Rightarrow \exp(-x) = \frac{1}{\exp(x)} \in (0,1)$.

now for part (i), we will cover it later.

see overleaf.

we define $e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.71828\ldots = \exp(1)$.

$\boxed{\text{Theorem}}$ (cont'd) $\forall x, y \in \mathbb{R}$, $\exp(x+y) = \exp(x)\cdot\exp(y)$.

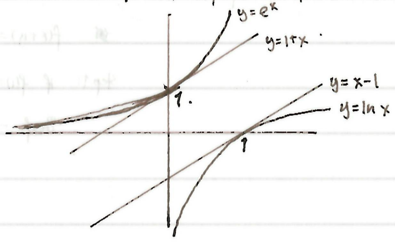
consequences of this theorem.

we know $\exp(0) = 1$, so if $x>0$, $\exp(x) > 1$. $\Rightarrow$ $\exp(x-x) = 1 = \exp(x)\exp(-x)$ $\Rightarrow$ if $x<0$ then $\exp(x) = \frac{1}{\exp(-x)} \in (0,1)$.

therefore $\exp(x) > 0$ $\forall x$ and $\exp(x) > 1 \iff x>0$, $\exp(x) < 1 \iff x<0$.

① $\cdot$ $\exp(x)$ is continuous, and uniformly continuous on all bounded subsets; and $\exp(x) = [\exp(1)]^x = e^x$.

② $\cdot$ $\forall x \in \mathbb{R}$, $e^x \geq 1+x$.



we can use this to prove $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

i.e. $\lim_{h\to 0} \frac{e^{0+h} - e^0}{h} = \frac{d}{dx} e^x \big|_{x=0}$.

③ $\cdot$ $\exp: \mathbb{R} \to (0,\infty)$ is bijective $\Rightarrow$ $\exists$ inverse function:

defined as <u>natural logarithm</u> $\ln: (0,\infty) \to \mathbb{R}$.

s.t. $\ln(e^x) = e^{\ln x} = x$.

using $e^x \geq 1+x$ and $\lim_{x\to 0}\frac{e^x-1}{x} = 1$, one can show that, for $\ln$,

- $\ln x \leq x-1$ $\forall x>0$ and $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$.
- now $\frac{\ln(1+x)}{x} \to 1$ as $x \to 0$ $\Rightarrow$ since exp is continuous, $\exp\left(\frac{\ln(1+x)}{x}\right) \to \exp(1) = e$ as $x\to 0$.

$e^{\ln(1+x)/x} = [e^{\ln(1+x)}]^{\frac{1}{x}} = (1+x)^{\frac{1}{x}}$ $\Rightarrow$ $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

$\therefore$ since $\frac{1}{n} \to 0$ as $n\to\infty$, $\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$ (replacing x by $\frac{1}{n}$).

Proof — we first use a lemma:

lemma — BINOMIAL THEOREM.

$\forall x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$\exp(x)\cdot\exp(y) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right)$

Distribute and group together all products with the same total degree:

$$\exp(x)\cdot\exp(y) = 1 + (x+y) + \left(\frac{y^2}{2!} + xy + \frac{x^2}{2!}\right) + \left(\frac{x^3}{3!} + \frac{x^2y}{2!} + \frac{xy^2}{2!} + \frac{y^3}{3!}\right) + \cdots$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^k y^{n-k}}{k!(n-k)!}$$

[is this analytically sound? question of convergence.].

we compare this to $\exp(x+y) = \sum_{n=0}^{\infty}\frac{(x+y)^n}{n!} \overset{\text{by the lemma.}}{=} \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}x^k y^{n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^k y^{n-k}}{k!(n-k)!} = \exp(x)\cdot\exp(y)$ // q.e.d.

Note however, that we need to address the question of convergence to make this precise.

in order to think of an infinite series in finite terms, we consider the behaviour of partial sums.

for $N \in \mathbb{N}$, we define $X_N = \sum_{n=0}^{N}\frac{x^n}{n!}$, $Y_N = \sum_{n=0}^{N}\frac{y^n}{n!}$, $Z_n = \sum_{n=0}^{N}\frac{(x+y)^n}{n!}$

as $N\to\infty$, $X_N \to \exp(x)$, $Y_N \to \exp(y)$, $Z_n \to \exp(x+y)$. then $X_N Y_N \to \exp(x)\exp(y)$ (combination thm).

we want to show that this matches $\lim_{N\to\infty} Z_n$;

or equivalently, $X_N Y_N - Z_N \to 0$. $X_N Y_N - Z_N = \left(1 + x + \cdots + \frac{x^N}{N!}\right)\left(1 + y + \cdots + \frac{y^N}{N!}\right) - \left[1 + (x+y) + \cdots + \frac{(x+y)^N}{N!}\right]$.

= sum of products of total degree $> N$ appearing in binomial expansion of $\sum_{n=0}^{\infty}\frac{(x+y)^n}{n}$

$\Rightarrow$ $|X_N Y_N - Z_N| \leq \sum_{n=N+1}^{\infty}\frac{(x+y)^n}{n!} \to 0$ as $N\to\infty$ //. q.e.d.

END OF SYLLABUS.